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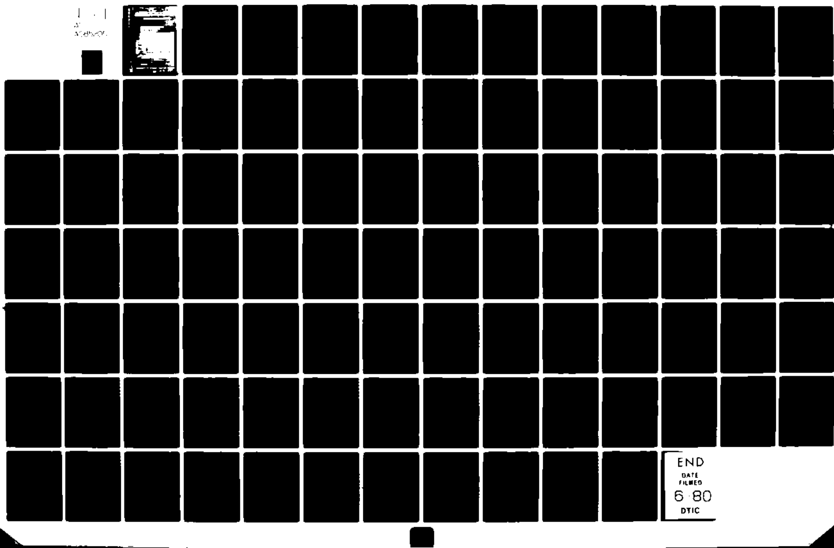

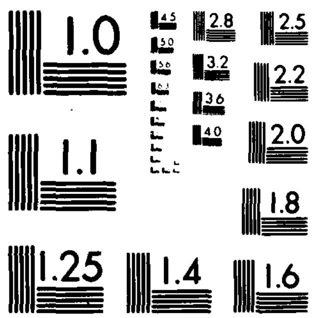

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A sampled data Stackelberg coordination scheme is developed in which both conceptual and computational issues are dealt with. Conditions which are sufficient for guaranteeing the existence of a stabilizing Stackelberg equilibrium solution are derived for a simple form of the Stackelberg strategy. Also, the role that the information structure can play in the decentralized control

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20. ABSTRACT (continued)

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DECENTRALIZED CONTROL AND MULTICRITERION DECISION MAKING

by

Peter Michael Walsh

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DECENTRALIZED CONTROL AND MULTICRITERION DECISION MAKING

BY

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B.S., Tufts University, 1974
M.S., University of Illinois, 1976

THESIS

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Electrical Engineering
in the Graduate College of the
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DECENTRALIZED CONTROL AND MULTICRITERION DECISION MAKING

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University of Illinois at Urbana-Champaign, 1980

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CHAPTER 1

INTRODUCTION

The design of a decentralized control scheme for a large scale system will, in the formulation of the problem, generally take advantage of some aspect of the structure of the system [16,17,18] or the problem may be formulated so that a desired structure is imposed on the system, e.g., [19,20,21,12].

Controller are generally implemented in some form of state feedback. Since many aspects of the system structure are variant under the control actions, the dependence of the decentralized control schemes on the structure maker many of the schemes crucially dependent on a uniformity of the goals of the individual decision makers.

In many situations, the individual decision makers will have different goals and it may be infeasible to have cooperation in agreeing on a common, single goal. So, given that there may exist multiple goals, it is of interest to analyze the decentralized control problem in this setting.

We will examine issues arising from the presence of conflicting goals among the decentralized controllers. The rational behavior of the controller is characterized by a strategy defining the rules of their behavior. Of primary interest to us will be the role that the Stackelberg strategy can play in the decentralized control problem and a number of conceptual issues that arise in attempting to make use of the strategy.

The Stackelberg strategy is well suited for use in designing a coordination scheme where there are many controllers acting on the system, each with a different criterion to be optimized. There are, however, some issues regarding the strategy which have yet to be resolved. Among these are the fact that the principle of optimality does not in general hold and its imposition for the continuous time case has yet to be satisfactorily dealt with. Also, unlike the classic single criterion linear quadratic problem, a closed form solution satisfying the necessary conditions or a satisfactory numerical solution technique have yet to be developed. In Chapter 2 we develop a sampled data equilibrium strategy which provides a computationally tractable solution technique.

This coordination technique is prescriptive, i.e., if a solution exists, it provides the methodology for calculating it. The existence of a solution is not assured. In an effort to establish conditions under which we can insure the existence of a solution satisfying the Stackelberg strategy, Chapter 3 will examine a very basic form of the Stackelberg strategy for dynamic games and sufficient conditions for the existence of a stabilizing solution will be developed. We restrict our attention to a formulation dealing with a linear continuous time system and in which the control laws are constrained to be linear state feedback. For this class of problems we are able to rely on the concepts of linear algebra to analyze the interaction of the individual decision maker's controllable and observable subspaces. By so doing we establish sufficient conditions under which the existence of a stabilizing solution can be assured.

Another form of the Stackelberg strategy will be seen in the remaining chapters, entering into the design of an information structure. The problem considered is one in which there are many controllers acting on a system where each controller has a different objective and their controls are determined according to the Nash equilibrium strategy. An example demonstrating the impact of the information structure is considered in Chapter 4. This is an example of a situation in which the availability of more information to one of the controllers has the effect of making that controller worse off. The demonstrated impact of the information structure in the example serves as motivation for the information structure design scheme of the next chapter.

In Chapter 5 we consider the design of an improved information structure by a somewhat unsuspected use of the Stabkelberg strategy. An iterative procedure is developed by which the information structure is altered to improve the overall system performance. The advantages inherent in the precedence nature of decision making under the Stackelberg strategy will be seen in this formulation.

CHAPTER 2

SAMPLED DATA EQUILIBRIUM STACKELBERG COORDINATION

2.1. Introduction

In this section, we consider the problem of formulating a hierarchical control structure for a multicontroller problem using the differential game concept of an equilibrium Stackelberg strategy. It is assumed that in general each agent has a different objective function and that one agent, the coordinator and Stackelberg leader, has an overall objective function.

There have been numerous investigations recently into the usefulness and characteristics of the Stackelberg strategy applied to dynamic systems [1-11]. In particular, the use of the Stackelberg strategy for the coordination of many agents has been considered in [4] and [11].

A form of periodic coordination has been considered by Chong and Athans [12] in which the vertical communication in the hierarchy is constrained to be periodic. Our basic assumptions are different from those of [12] and subsequently the nature of the solutions are quite dissimilar.

With a Stackelberg strategy, we assure it is known that one player, the coordinator and Stackelberg leader, will determine his controls before any of the other players (followers or lower level decisionmakers). The lower level decisionmakers then perform their optimization subject to their knowledge of the coordinator's decision, that is, they are reacting to his decision. The followers act simultaneously and we consider the case when they play a Nash strategy among themselves. The leader performs his optimization subject to the expected reactions of the followers. The leader's ability to make decisions first, taking into account the reactions of the lower level decisionmakers, enables him, to a degree, to impose his criterion onto the other controllers.

This strategy is appropriate for imposing a control structure on a problem in which there are many decision makers with different criteria unable or unwilling to cooperate in their decision making process and in which a hierarchy of decision making already exists or can be imposed.

The solution of the closed loop Stackelberg problem generally depends on the length of the interval over which the problem is defined as well as the state of the system at the initial time [8]. Implicit in the solution is a guarantee by the leader that he will not deviate from his announced control rule. If the problem is redefined on a subinterval of the time interval of the original problem, the solutions on this interval would in general be different. Thus, the principle of optimality does not, in general, hold.

The feedback Stackelberg strategy is defined as a closed loop Stackelberg strategy which has the added constraint that the leader's control is required to satisfy the principle of optimality [8]. Generalization to equilibrium Stackelberg strategies is introduced in [7]. Further discussion of the Nash and Stackelberg strategies for dynamic games can be found in the references.

The open loop, closed loop, feedback and sampled data Stackelberg strategies exhibit notably different characteristics due to the fact that they are based on fundamentally different problem formulations. In order to see the motivation and significance of the sampled data formulation it is necessary to appreciate two particular aspects of the continuous time Stackelberg problem.

First, as described above, the solution of the closed loop Stackelberg problem for dynamic games does not, in general, satisfy the principle of optimality. The imposition of the principle of optimality for discrete time games has been considered in [8] while the procedure for doing this for continuous time games has yet to be resolved.

A second peculiarity of the closed loop Stackelberg problem is that, unlike the classic single agent, linear quadratic control problem, or even certain multicontroller problems, the necessary conditions derived by the variational technique for the linear quadratic, continuous time, closed loop Stackelberg problem result in a nonlinear control, the existence of which is not assured [6], [16].

With these aspects of the continuous time Stackelberg problem in mind, the significance of the sampled data formulation is apparent. That is, the resultant control laws are piecewise continuous linear time varying functions of the measurements for the linear quadratic case and, as we have formulated it, the principle of optimality holds at the sampling times.

Recent work on the Stackelberg strategy for continuous time dynamic systems has concentrated primarily on the open loop formulation [4] and on the linearly constrained closed loop formulation [6]. For the linear quadratic case, the open loop solution is a linear function of the initial condition and the solution in [6] is linear by construction but the principle of optimality does not, in general, hold. The linear form of the sampled data solution is a direct result of this information constraint and is not due to any structural (linear) constraint being imposed on the form of the solution.

By considering the sampled data formulation we have been able to obtain a responsive state feedback solution, which is tractable, has a very simple form for implementation, and for which the principle of optimality holds at the sampling times. Of equal importance is the existence of an efficient algorithm for the calculation of this solution. In this chapter we derive a computationally efficient technique for obtaining the solution for the linear quadratic sampled data equilibrium Stackelberg strategy. The solution algorithm tends (i) to minimize the on-line computations and (ii) to take advantage of the nature of the sampled data solution to greatly reduce the horizon over which integrations must be performed, thereby reducing off-line computations as well. These features are obtained as a result of employing a form of invariant imbedding [13].

In Section 2.2 we formulate the problem and present necessary conditions for the solution. The linear quadratic case will be considered in Section 2.3 and techniques for the solution of the linear quadratic case will be discussed in Section 2.4. Section 2.5 summarizes the results.

2.2. Sampled Data Equilibrium Stackelberg Formulation

Consider the system

$$\dot{x} = f(x, u_i; i = 0, 1, \ldots, m), \quad x(t_o) = x_o, \tag{2.1}$$

$u_i \in R^{r_i}$, $x \in R^n$, where r_i is the dimension of the ith control vector. Each lower level control, u_i, for $i = 1, \ldots, m$, is chosen to reduce as much as possible the scalar index

$$J_i = K_{if}(x(t_f)) + \int_{t_o}^{t_f} L_i(x, u_j; j = 0,1,\ldots,m)dt. \tag{2.2}$$

The coordinator's control, u_o, is chosen to reduce as much as possible the scalar index

$$J_o = K_{of}(x(t_f)) + \int_{t_o}^{t_f} L_o(x, u_i; i = 0,1,\ldots,m)dt. \tag{2.3}$$

The terminal time, t_f, is fixed.

The information is assumed to be in the form of sampled data acquisition, that is, measurements are taken at r discrete instances in time $\{t_i \in [t_o, t_f), i = 0,1,\ldots,r-1\}$. The controls will be functions of time and the latest state measurement, i.e., $u_i = u_i(t, x_j)$ for $t_j \le t < t_{j+1}$, for all i, where $x_j \triangleq x(t_j)$.

The leader will calculate and announce $u_o(t, x_j)$ for $t \in [t_j, t_{j+1})$, and $j = 0,1,\ldots,r-1$ at the beginning of the game. This control is chosen to minimize the leader's performance index under the assumption that the followers will in turn be minimizing their respective performance indices subject to the announced leader's control, and subject to the requirement that the leader's control remains optimal for any game starting at t_j, $j = 0,1,\ldots,r-1$. The controls are calculated based on the assumption that future measurements will be available at t_k, $k = j+1,\ldots,r-1$.

In contrast to the single controller case or even certain multi-controller strategies, the Stackelberg controls including sampled data closed loop control do not in general satisfy the principle of optimality [8]. In this section we derive necessary conditions for <u>sampled data</u>

equilibrium Stackelberg strategies whereby the principle of optimality is imposed at the sampling times t_j, $j = 0,1,\ldots,r-1$.

Let the optimum costs to go at time t_j be denoted by $V_i^*(x(t_j),t_j)$, $i = 0,1,\ldots,m$. Imposing the principle of optimality we have

$$V_i^*(x_j,t_j) = \min_{u_i}\{V_i^*(x_{j+1},t_{j+1}) + \int_{t_j}^{t_{j+1}} L_i(x,u_k;k=0,1,\ldots,m)dt\} \qquad (2.4)$$

where

$$V_i^*(x(t_f),t_f) = K_{if}(x(t_f)), \quad i = 0,1,\ldots,m \qquad (2.5)$$

and where the minimization with respect to u_i in (2.4) is subject to the system constraint and to the minimization being performed by the other controllers according to the strategy outlined in the preceding paragraphs. Note that the optimizations of the future periods are imbedded in the term $V_i^*(x_{j+1},t_{j+1})$. Also notice that at sample time t_j, all controls from t_j through t_f will, in principle, be calculated and that they are independent of any control action prior to t_j except for the effect of x_j. So, by the nature of the problem formulation, the solution will satisfy the principle of optimality at the sampling times. This aspect of the sampled data formulation is analogous to the feedback formulations of [8], or the equilibrium formulation [7].

The variational method is applied to (2.1), (2.4), and (2.5), to obtain the necessary conditions. These conditions are an extension of those derived in [11]. The necessary conditions for the followers on $[t_j,t_{j+1})$ for $i = 1,\ldots,m$ are

$$\dot{x} = f(x,u_i;\ i = 0,1,\ldots,m),\ x(t_j) = x_j \qquad (2.6)$$

$$\dot{p}_i = -\frac{\partial H_i'}{\partial x}, \quad p_i(t_{j+1})' = \frac{\partial V_i^*(x(t_{j+1}), t_{j+1})}{\partial x(t_{j+1})} \tag{2.7}$$

$$0 = \frac{\partial H_i}{\partial u_i} \tag{2.8}$$

where

$$H_i(x, p_i, u_k; k=0,1,\dots,m) = L_i(x, u_k; k=0,1,\dots,m) + p_i' f(x, u_k; k=0,1,\dots,m). \tag{2.9}$$

The necessary conditions for the leader on $[t_j, t_{j+1})$ are

$$\dot{\lambda} = -\frac{\partial H_o'}{\partial x}, \quad \lambda(t_{j+1})' = \frac{\partial V_o^*(x(t_{j+1}), t_{j+1})}{\partial x(t_{j+1})} - \sum_{k=1}^{m} \gamma_k'(t_{j+1}^-) \frac{\partial^2 V_k^*(x(t_{j+1}), t_{j+1})}{\partial x(t_{j+1})^2} \tag{2.10}$$

$$\dot{\gamma}_i = -\frac{\partial H_o'}{\partial p_i}, \quad \gamma_i(t_j^+) = 0, \quad i = 1,\dots,m \tag{2.11}$$

where $\gamma_i(t_j^-) = \lim_{t \to t_j^-} \gamma_i(t)$ for γ_i defined on the (j-1)st interval $[t_{j-1}, t_j)$ $\gamma_i(t_j^+) = \gamma_i(t_j)$ defined on the jth interval $[t_j, t_{j+1})$.

$$\frac{\partial H_o}{\partial u_o} = 0 \tag{2.12}$$

$$\frac{\partial H_o}{\partial u_i} = 0, \quad i = 1,\dots,m \tag{2.13}$$

where

$$H_o(x, \lambda, p_i, \gamma_i, \beta_i;\ i=1,2,\dots,m, u_j;\ j=0,1,\dots,m) = L_o(x, u_i;\ i=0,1,\dots,m)$$
$$+ \lambda' f(x, u_i;\ i=0,1,\dots,m) + \sum_{k=1}^{m} \{\gamma_k'(-\frac{\partial H_k}{\partial x})' + \beta_k'(\frac{\partial H_k}{\partial u_k})'\}. \tag{2.14}$$

Equation (2.13) and the constraints appended under the summation sign in (2.14) are due to the leader taking into account the reactions of the lower level decisionmakers. The solution conditions on the β_k are implicit in equation (2.13).

2.3. The Linear Quadratic Case

Assume the system is linear

$$\dot{x} = Ax + \sum_{i=0}^{m} B_i u_i \tag{2.15}$$

$$x(t_o) = x_o \tag{2.16}$$

and the criteria quadratic

$$J_i = \frac{1}{2} x' K_{if} x \Big|_{t=t_f} + \frac{1}{2} \int_{t_o}^{t_f} (x' Q_i x + \sum_{j=0}^{m} u_j' R_{i,j} u_j) dt. \tag{2.17}$$

The necessary conditions for the lower level controllers for $t \in [t_j, t_{j+1})$ and $i = 1, \ldots, m$ are

$$\dot{p}_i = -Q_i x - A' p_i, \quad p_i'(t_{j+1}) = \frac{\partial V_i^*(x(t_{j+1}), t_{j+1})}{\partial x(t_{j+1})} \tag{2.18}$$

$$u_i = -R_{i,i}^{-1} B_i' p_i. \tag{2.19}$$

The necessary conditions for the leader are

$$\begin{aligned} \dot{\lambda} &= -Q_o x - A'\lambda + \sum_{i=1}^{m} Q_i \gamma_i \\ \lambda'(t_{j+1}) &= \frac{\partial V_o^*(x(t_{j+1}), t_{j+1})}{\partial x(t_{j+1})} - \sum_{i=1}^{m} \gamma_i'(t_{j+1}^-) \frac{\partial^2 V_i^*(x(t_{j+1}), t_{j+1})}{\partial x(t_{j+1})^2} \end{aligned} \tag{2.20}$$

$$\dot{\gamma}_i = A\gamma_i - S_{o,i} p_i + S_i \lambda, \quad \gamma_i(t_j^+) = 0 \tag{2.21}$$

$$u_o = -R_{o,o}^{-1} B_o' \lambda \tag{2.22}$$

where

$$S_i \triangleq B_i R_{i,i}^{-1} B_i'$$

$$S_{j,i} \triangleq B_i R_{i,i}^{-1} R_{j,i} R_{i,i}^{-1} B'.$$

During each interval, the state will evolve according to

$$\dot{x} = Ax - \sum_{i=1}^{m} S_i p_i - S_o \lambda \tag{2.23}$$

for $t \in [t_j, t_{j+1})$ where $x(t_j)$ is determined in the previous interval.

If the state measurements are made at r discrete instances in time, we are faced with an (r+1)-point boundary value problem. At this stage, there are two alternate approaches we can take to the problem. The first and standard approach starts by assuming an explicit functional dependence of the costates on the state. This results in a set of coupled matrix Riccati equations which must be solved repeatedly at each sample time. A general algorithm for the efficient solution of these equations for each new set of boundary conditions will be outlined in the next section. We will also consider an even more efficient approach utilizing invariant imbedding [13,14]. It is based on an assumption of the functional dependence of the state and costates on one another and of their explicit dependence on their respective boundary conditions. This result will be shown in detail.

2.4. Solution of the Linear Quadratic Problem

The first approach to dealing with the r+1 point boundary value problem starts by assuming that the costates depend on the states by affine functions. The affine dependence, rather than simply linear, is necessary so that the lower level decisionmakers will be able to calculate their controls as functions of the leader's announced control, i.e., their computations will be coupled to the leader's sequentially, not simultaneously.

Differential equations can be found for the coefficients of these functions and for the associated costs to go. If m is the number of controllers, the problem can be reduced to that of solving m coupled matrix Riccati equations and m matrix Lyapunov equations at each sample time, all with boundary conditions at a common time. The same set of equations are resolved at each sample time with only a change in the boundary conditions. A sampled data Nash formulation has been considered by Simaan and Cruz [9] and a computational technique for the solution of the resultant Riccati equations has also been obtained [10]. We have obtained a generalization of [10] in which the solutions of the Riccati equations are expressed in terms of a preliminary solution due to a specific set of boundary conditions and a correction term dependent on the actual boundary conditions. An algorithm is found for finding these correction terms requiring the solution of m uncoupled matrix Riccati equations, thus providing substantial improvement over a brute force solution of the coupled equations.

With the first technique, we assume that the cost to go functions are of the form

$$V_{1i}(t) = \left(\tfrac{1}{2}x'E_{1i}x + e_{1i}'x + q_{1i}\right)\Big|_t$$

$$V_2(t) = \left(\tfrac{1}{2}x'E_2x + e_2'x + q_2\right)\Big|_t$$

in which case, the boundary conditions in (18) and (20) are

$$p_i(t_{j+1}) = (E_{1i}x + e_{1i})\Big|_{t_{j+1}}$$

and

$$\lambda(t_{j+1}) = (E_2 x + e_2 - \sum_{i=1}^{m} E_{1i}\gamma_i)\Big|_{t_{j+1}}$$

As is conventionally done in solving the two point boundary value problems in optimal control, we assume a functional dependence of the costates on the state. An affine dependence is assumed due to the nature of the Stackelberg problem. Thus we assume

$$\begin{aligned} p_i &= K_{1i}x + g_{1i} \\ \gamma_i &= K_{3i}x + g_{3i} \\ \lambda &= K_2 x + g_2 . \end{aligned} \tag{2.24}$$

By differentiation of these equations and from the equations of section 3, we find that the K matrices and g vectors must satisfy

$$\dot{K}_{1i} = -Q_{1i} - A'K_{1i} - K_{1i}A + K_{1i}[\sum_{j=1}^{m} S_{1j}K_{1j} + S_2K_2], K_{1i}(t_{j+1}) = E_{1i}(t_{j+1})$$

$$\dot{g}_{1i} = -A'g_{1i} + K_{1i}[\sum_{j=1}^{m} S_{1j}g_{1j} + S_2g_2], g_{1i}(t_{j+1}) = p_{1i}(t_{j+1})$$

$$\dot{K}_2 = -Q_2 - A'K_2 - K_2A + \sum_{i=1}^{m} Q_{1i}K_{3i} + K_2 \sum_{i=1}^{m} S_{1i}K_{1i} + K_2S_2K_2$$

$$K_2(t_{j+1}) = P_2(t_{j+1}) - \sum_{i=1}^{m} P_{1i}(t_{j+1})K_{3i}(t_{j+1}^-)$$

$$\dot{g}_2 = -A'g_2 + K_2 \sum_{i=1}^{m} S_{1i}g_{1i} + K_2S_2g_2 + \sum_{i=1}^{m} Q_{1i}g_{3i}$$

$$g_2(t_{j+1}) = p_2(t_{j+1}) - \sum_{i=1}^{m} P_{1i}(t_{j+1})g_{3i}(t_{j+1}^-)$$

$$\dot{K}_{3i} = AK_{3i} - K_{3i}A - S_{2,1i}K_{1i} + S_{1i}K_2 + K_{3i} \sum_{j=1}^{m} S_{1j}K_{1j} + K_{3i}S_2K_2$$

$$\dot{g}_{3i} = Ag_{3i} - S_{2,1i} + S_{1i}g_2 + K_{3i}[\sum_{j=1}^{m} S_{1j}g_{1j} + S_2g_2]$$

$$K_{3i}(t_j^+) = 0, \qquad g_{3i}(t^+_j) = 0 \tag{2.25}$$

The coefficients in the cost to go functions must, in each interval, satisfy the following equations

$$
\begin{aligned}
\dot{E}_{1i} &= -E_{1i}\tilde{A} - \tilde{A}'E_{1i} - N^{1}_{1i} \\
\dot{e}_{1i} &= -\tilde{A}'e_{1i} + 2E_{1i}[\sum_{j=1}^{m} S_{1j}g_{1j} + S_2 g_2] - \frac{1}{2}N^{2}_{1i} \\
\dot{q}_{1i} &= (\sum_{j=1}^{m} g'_{1j}S_{1j} + g'_2 S_2)p_{1i} - \frac{1}{2}N^{3}_{1i} \\
& E_{1i}(t_f) = K_{1f}, \quad e_{1i}(t_f) = 0, \quad q_{1i}(t_f) = 0 \\
& E_{1i}(t^-_{j+1}) = E_{1i}(t^+_{j+1}), \quad e_{1i}(t^-_{j+1}) = e_{1i}(t^+_{j+1}), \\
& \quad q_{1i}(t^-_{j+1}) = q_{1i}(t^+_{j+1})
\end{aligned}
\tag{2.26}
$$

and the $\tilde{A}$, N^{j}_{1i}, $j = 1,2,3$ are known in terms of the previous solution of (2.25). Equations of the same form are also satisfied by the coefficients E_2, e_2, and q_2. The assumed dependence of the costates on the state, equation (2.25), results in a set of equations which, unlike the conventional optimal control problem, are themselves a two point boundary value problem. This is a result of the leader appending the two point boundary value problem which results from the followers' optimizations. So, we assume an explicit dependence of the solution of the K_{3i} and the g_{3i} on the solution of the K_{1i}, g_{1i}, K_2 and g_2 in order to reduce these equations to a solvable single point boundary value problem. Thus we assume

$$
\begin{aligned}
K_{3i} &= F_{3i,2}K_2 + \sum_{j=1}^{m} F_{3i,1j}K_{1j} + F_{3i,4} \\
g_{3i} &= F_{3i,2}g_2 + \sum_{j=1}^{m} F_{3i,1j}g_{1j}
\end{aligned}
$$

Notice that the same coefficient matrices appear in both equations. This, it turns out, is sufficient to obtain the desired dependence. The differential equations that these coefficient matrices must satisfy are

$$\dot{F}_{3i,2} = AF_{3i,2} + F_{3i,2}A' + S_{1i} + F_{3i,4}S_2 - F_{3i,2}\sum_{j=1}^{m} Q_{1j}F_{3j,2}, F_{3i,2}(t_j^+) = 0$$

$$\dot{F}_{3i,1k} = AF_{3i,1k} + F_{3i,1k}A' + F_{3i,4}S_{1k} - F_{3i,2}\sum_{j=1}^{m} Q_{1j}F_{3i,1k} \quad i \neq k$$

$$\dot{F}_{3i,1k} = AF_{3i,1k} + F_{3i,1k}A' + F_{3i,4}S_{1k} - F_{3i,2}\sum_{j=1}^{m} Q_{1j}F_{3i,1k} - S_{2,1k}, \quad i = k$$

$$F_{3i,1k}(t_j^+) = 0$$

$$\dot{F}_{3i,4} = AF_{3i,4} + F_{3i,4}A' + F_{3i,2}Q_2 - F_{3i,2}\sum_{j=1}^{m} (Q_{1j}F_{3j,4} + F_{3i,1j}Q_{1j})F_{3i,4}(t_j^+) = 0$$

These equations are solved once only, and for a period of one sample interval. The solution is then used repeatedly during each sample interval, plugging into equations (2.25), converting (2.25) to a single point boundary value problem in which only the boundary condition changes between sample intervals.

Having done this, we are now at a point where the solution is expressed entirely in terms of equations of the following general form of the coupled matrix Riccati equations

$$\dot{K}_j^\ell = A_{1,j}^\ell K_j^\ell + K_j^\ell A_{2,j}^\ell + \sum_{k=1}^{\ell} T_{j,k}^\ell K_k^\ell + K_j^\ell \sum_{k=1}^{\ell} D_{j,k}^\ell K_k^\ell + Q_j^\ell \tag{R}$$

$$K_j^\ell(T_{i+1}); \text{ known} \qquad j = 1, \ldots, \ell$$

where ℓ is the number of coupled equations. These must be solved repeatedly with changes occuring only in the boundary conditions.

The final step for this technique of solving the sampled data problem is to derive an efficient technique for the repeated solution of coupled equations of the form (R). What follows is a generalization of [10] and [15].

The approach taken is to express the solutions K_j^ℓ, corresponding to the actual boundary conditions, in terms of a solution $\hat{K}_j^\ell$ which corresponds to some other arbitrary, known boundary conditions. Equations for the correction terms are found and a sequence of steps leads to a solution which requires ℓ uncoupled matrix Riccati equations to be solved each time and some auxilary equations must be solved once only and over a period equal to the sample interval. The details follow.

Define

$$Z_j^\ell = [K_j^\ell - \hat{K}_j^\ell]^{-1}$$

then

$$K_j^\ell = \hat{K}_j^\ell + (Z_j^\ell)^{-1}$$

Differentiating, we find

$$\dot{Z}_j^\ell = -Z_j^\ell A_{1,j}^\ell + \bar{A}_j^\ell Z_j^\ell + Z_j^\ell \sum_{k=1}^{\ell} G_{j,k}^\ell (Z_k^\ell)^{-1} Z_j^\ell + \sum_{k=1}^{\ell} H_{j,k}^\ell (Z_k^\ell)^{-1} Z_j^\ell$$

$$Z_j^\ell(t_{k+1}) = [K_j^\ell(t_{k+1}) - \hat{K}_j^\ell(t_{k+1})]^{-1} \qquad j = 1,2,\ldots,\ell$$

where

$$\bar{A}_j^\ell \triangleq (-A_{2,j}^\ell - \sum_{k=1}^{\ell} D_{j,k}^\ell \hat{K}_k^\ell)$$

$$G_{j,k}^\ell \triangleq (-\hat{K}_j^\ell D_{j,k}^\ell - T_{j,k}^\ell)$$

$$H_{j,k}^\ell \triangleq -D_{j,k}^\ell$$

If we define the term

$$K_j^{\ell-1} \triangleq (Z_j^\ell)^{-1} Z_\ell^\ell , \qquad j = 1,2,\ldots,\ell-1$$

differentiation yields

$$\dot{K}_j^{\ell-1} = A_{1,j}^{\ell-1}K_j^{\ell-1} + K_j^{\ell-1}A_{2,j}^{\ell-1} + \sum_{k=1}^{\ell-1} T_{j,k}^{\ell-1}K_k^{\ell-1} + K_j^{\ell-1}\sum_{k=1}^{\ell-1} D_{j,k}^{\ell-1}K_k^{\ell-1} + Q_j^{\ell-1} \;;$$

$$K_j^{\ell-1}(t_{k+1}) = [Z_j^{\ell}(t_{k+1})]^{-1}Z_{\ell}^{\ell}(t_{k+1}) \quad j = 1,\ldots,\ell-1.$$

where

$$A_{1,j}^{\ell-1} = A_{1j}^{\ell}$$

$$A_{2,j}^{\ell-1} = (-A_1^{\ell} + G_{j,\ell}^{\ell})$$

$$T_{j,k}^{\ell-1} = -G_{j,k}^{\ell}$$

$$D_{j,k}^{\ell-1} = +G_{j,k}^{\ell}$$

$$Q_j^{\ell-1} = -G_{j,\ell}^{\ell}$$

So, the solution of the Z_j^{ℓ} equations can be obtained once the $K_j^{\ell-1}$ equations are solved. However, the solutions for the original equations, K_j^{ℓ} are expressed in terms the $(Z_j^{\ell})^{-1}$ $j=1,\ldots,\ell$ and, given $(Z_{\ell}^{\ell})^{-1}$, the remaining $(Z_j^{\ell})^{-1}$ for $j=1,\ldots,\ell-1$ are known in terms of $(Z_{\ell}^{\ell})^{-1}$ and the $K_j^{\ell-1}$, $j=1,\ldots,\ell-1$. For each i, $i=1,\ldots,\ell$, the term $(Z_i^i)^{-1}$ satisfies

$$(\dot{Z}_i^i)^{-1} = (A_{1,j}^i - \sum_{k=1}^{i} G_{i,k}^i K_k^{i-1})(Z_i^i)^{-1} - (Z_i^i)^{-1}\hat{A}_i^i - (Z_i^i)^{-1}[\sum_{k=1}^{i} H_{i,k}^i K_k^{i-1}]\,(Z_i^i)^{-1}$$

$$(Z_i^i(t_{j+1}))^{-1} = [K_i^i(t_{j+1}) - \hat{K}_i^i(t_{j+1})]$$

Thus, we have the K_j^{ℓ}, $j=1,\ldots,\ell$ known in terms of $(Z_{\ell}^{\ell})^{-1}$ and the $K_j^{\ell-1}$, $j=1,\ldots,\ell-1$. The equations in $K_j^{\ell-1}$ are of the same form as the equations in K_j^{ℓ} and so the technique is applied again and is done recursively until we reach K_1^1. Notice that at level i of the recursion only one equation, in

$(Z_i^i)^{-1}$, need be integrated.

In summary, once the preliminary solutions, $\hat{K}_i^j$, are found, the desired solutions K_i^j are obtained as follows;

$$
\begin{array}{ll}
\text{solve for } (Z_1^1)^{-1} & \\
K_1^1 = \hat{K}_1^1 + (Z_1^1)^{-1} & \\
\hline
\text{solve for } (Z_2^2)^{-1} & \\
(Z_1^2)^{-1} = K_1^1 (Z_2^2)^{-1} & \\
K_2^2 = \hat{K}_2^2 + (Z_2^2)^{-1} & \\
K_1^2 = \hat{K}_1^2 + (Z_1^2)^{-1} & \\
\hline
\text{solve for } (Z_3^3)^{-1} & \\
(Z_1^3)^{-1} = K_1^2 (Z_3^3)^{-1} & \\
(Z_2^3)^{-1} = K_2^2 (Z_3^3)^{-1} & \\
K_3^3 = \hat{K}_3^3 + (Z_3^3)^{-1} & \\
K_2^3 = \hat{K}_2^3 + (Z_2^3)^{-1} & \\
K_1^3 = \hat{K}_1^3 + (Z_1^3)^{-1} & \\
\hline
\vdots & \\
\hline
\text{solve for } (Z_\ell^\ell)^{-1} & \\
(Z_j^\ell)^{-1} = K_j^{\ell-1} (Z_\ell^\ell)^{-1} & j=1,\ldots,\ell-1 \\
K_j^\ell = \hat{K}_j^\ell + (Z_j^\ell)^{-1} & j=1,\ldots,\ell
\end{array}
$$

So, in this first approach, the sampled data Stackelberg problem is reduced to the solution of a set of ℓ coupled matrix Riccati equations, each of dimension n, which must be solved at each sample time with only a change in the boundary conditions. The solution of these equations for any boundary condition is then found to be expressible in terms of the solution of an auxiliary problem. The needed correction terms require the solution of ℓ uncoupled matrix Riccati equations of dimension n, at each sample time. Considerable savings in computation will accrue if there are a large number of samples, which is typically the case.

2.4.1. The Second Approach: Invariant Imbedding

The ultimate goal when deriving the solution technique is to minimize the amount of computations required by taking advantage of the fact that the equations to be solved are the same in each sample interval and only the boundary conditions change.

The derivations performed in the remainder of this section will proceed as outlined below. First we define more compact notation, grouping the state and costates according to their boundary conditions. We then assume an explicit functional dependence of the costates on the state and on the costates' boundary conditions. Due to this assumption, the solutions of the resultant equations are independent of the changing costates' boundary conditions and it is because of this independence that we are able to obtain the computational savings. The cost to go equations are derived since they are needed to generate the appropriate boundary conditions to plug into the solution functions. A functional dependence of the costs to go on their boundary conditions is also assumed and finally the boundary conditions for each interval are established in terms of those in the adjacent interval. The details of the derivation follow.

Rather than making the standard assumption of a functional dependence of the costates on the state alone as in the first approach, we will make a different assumption. Notice that on the interval $[t_j, t_{j+1})$, the costates p_{1i}, $\forall i$, equations (2.18) and λ, equation (2.20), have boundary conditions at t_{j+1}. The costates γ_i, $\forall i$, equations (2.21) and the state x, equation (2.23), have boundary conditions at t_j. For convenience of notation, let us group the state and costate vectors according to boundary conditions as follows

$$y_1 \triangleq x$$
$$y_2 \triangleq (\gamma_1' \,\vdots\, \gamma_2' \,\vdots\, \cdots \,\vdots\, \gamma_m')'$$
$$y_3 \triangleq (\lambda' \,\vdots\, p_1' \,\vdots\, p_2' \,\vdots\, \cdots \,\vdots\, p_m')'.$$

Now equations (2.18), (2.20), (2.21) and (2.23) can be expressed as

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & A_{13} \\ 0 & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \tag{2.27}$$

where the A_{ij} of (2.27) are appropriate concatenation of the Q, A and S matrices of (2.18), (2.20), (2.21) and (2.23). In each interval $[t_j, t_{j+1})$, the vectors y_1 and y_2 have boundary conditions at t_j and the vector y_3 has boundary conditions at t_{j+1}.

$$y_2(t_j^+) = 0 \tag{2.28}$$

$$y_3(t_{j+1}^-) = \begin{bmatrix} \lambda \\ --- \\ p_1 \\ \vdots \\ p_m \end{bmatrix} = \left.\begin{bmatrix} \dfrac{\partial V_o'}{\partial y_1} - \sum_{i=1}^{m} \dfrac{\partial^2 V_i}{\partial y_1^2} \cdot \gamma_i \\ ------------------ \\ \dfrac{\partial V_1'}{\partial y_1} \\ \vdots \\ \dfrac{\partial V_m'}{\partial y_1} \end{bmatrix}\right|_{t_{j+1}^-} \tag{2.29}$$

where $y_2(t_j^+) \triangleq y_2(t_j)$ defined on the interval $[t_j, t_{j+1})$ and $y_3(t_{j+1}^-) \triangleq \lim_{t \to t_{j+1}^-} y_3(t)$ for $y_3(t)$ defined on the interval $[t_j, t_{j+1})$.

It is in the next step where we deviate from the standard approach. We will make assumptions of the functional dependence of the costates on the state and on the costates' boundary conditions. In so doing we will be able to solve for these functions independent of the costates' boundary conditions.

For $t \in [t_j, t_{j+1})$ assume[1]

$$y_2(t) = F_1(t)y_1(t) + F_2(t)y_2(t_j) + F_3(t)y_3(t_{j+1}) \tag{2.30}$$

and

$$y_3(t) = G_1(t)y_1(t) + G_2(t)y_2(t) + G_3(t)y_3(t_{j+1}). \tag{2.31}$$

By differentiation of (2.30) and (2.31) and by substitution of (2.27), we find[2]

$$\dot{G}_1 = A_{31} + A_{33}G_1 - G_1A_{11} - G_1A_{13}G_1 - G_2A_{23}G_1, \quad G_1(t_{j+1}) = 0 \tag{2.32}$$

$$\dot{G}_2 = A_{32} + A_{33}G_2 - G_1A_{13}G_2 - G_2A_{22} - G_2A_{23}G_2, \quad G_2(t_{j+1}) = 0 \tag{2.33}$$

$$\dot{G}_3 = (A_{33} - G_1A_{13} - G_2A_{23})G_3, \quad G_3(t_{j+1}) = I \tag{2.34}$$

$$\dot{F}_1 = (A_{22} + A_{23}G_2)F_1 - F_1(A_{11} + A_{13}G_1) - F_1A_{13}G_2F_1 + A_{23}G_1, \quad F_1(t_j) = 0 \tag{2.35}$$

$$\dot{F}_2 = A_{22}F_2 + A_{23}G_2F_2 - F_1A_{13}G_2F_2, \quad F_2(t_j) = I \tag{2.36}$$

$$\dot{F}_3 = (A_{22} + A_{23}G_2 - F_1A_{13}G_2)F_3 + A_{23}G_3 - F_1A_{13}G_3, \quad F_3(t_j) = 0. \tag{2.37}$$

---

[1] The dependence of $y_3(t)$ on $y_2(t)$ instead of $y_2(t_j)$ results in simplified computations.

[2] All matrices are evaluated at time t unless indicated otherwise.

Since $y_2(t_j) = 0$ and by substituting (2.30) into (2.31) we have

$$y_2(t) = F_1(t)y_1(t) + F_3(t)y_3(t_{j+1}) \tag{2.38}$$

$$y_3(t) = \bar{G}_1(t)y_1(t) + \bar{G}_3(t)y_3(t_{j+1}) \tag{2.39}$$

where $\bar{G}_1' = G_1 + G_2F_1$ and $\bar{G}_3 = G_3 + G_2F_3$.

For $t \in [t_j, t_{j+1})$ assume[1]

$$y_1(t) = H_1(t)y_1(t_j) + H_3(t)y_3(t_{j+1}) \tag{2.40}$$

by differentiation of (2.40) and substitution of (2.27) and (2.39) we find

$$\dot{H}_1 = (A_{11} + A_{13}\bar{G}_1)H_1, \quad H_1(t_j) = I \tag{2.41}$$

$$\dot{H}_3 = (A_{11} + A_{13}\bar{G}_1)H_3 + A_{13}\bar{G}_3, \quad H_3(t_j) = 0. \tag{2.42}$$

If the system (2.15) and the criteria functions (2.17) are time invariant and if the sampling rate is constant, that is if $(t_{j+1} - t_j) = T = \text{constant}$ for all j, the equations (2.32) through (2.37), (2.41) and (2.42) will be the same for each interval. Then, since their boundary conditions are invariant, these equations will have to be solved only once and the same solution will be valid for every interval $[t_j, t_{j+1})$, $j = 0,1,\ldots,r-1$.

2.4.2. Boundary Conditions and Cost To Go Equations

The boundary conditions for the costate equations on the jth interval $[t_j, t_{j+1})$ are known in terms of the costs to go at the end of the interval, (2.7) and (2.10). Therefore, for the purpose of obtaining the costates' boundary conditions, we must first derive the cost to go equations. First, substituting (2.19) and (2.22) for the controls and with the form of

[1]Because (2.30) and (2.31) reduce to (2.38) and (2.39), the dependence of $y_1(t)$ on $y_2(t_j)$ need not be assumed.

the solution for y_3 as in (2.39), recalling that $y_3 \triangleq (\lambda' \vdots p_1' \vdots \cdots \vdots p_m')'$, the integrands L_i of the criterion functions can be written

$$L_i = \frac{1}{2}\{x'Q_i x + \sum_{j=0}^{m} u'_j R_{ij} u\} = \frac{1}{2}\{y_1'Q_i y_1 + y_3'\hat{S}_i y_3\}$$

and for $t \in [t_j, t_{j+1})$,

$$L_i = \frac{1}{2}\{y_1'\bar{S}_{i1}y_1 + y_3(t_{j+1})'\bar{S}_{i2}y_3(t_{j+1})\} + y_1'\bar{S}_{i3}y_3(t_{j+1}) \tag{2.43}$$

where all variables are evaluated at time t unless indicated otherwise, and where

$$\bar{S}_{i1} = Q_i + \bar{G}_1'\hat{S}_i\bar{G}_1$$

$$\bar{S}_{i2} = \bar{G}_3'\hat{S}_i\bar{G}_3$$

$$\bar{S}_{i3} = \bar{G}_1'S_i\bar{G}_3$$

and

$$\hat{S}_i \triangleq \begin{bmatrix} S_{io} & & & 0 \\ & S_{i1} & & \\ & & \ddots & \\ 0 & & & S_{im} \end{bmatrix}.$$

Due to the assumed explicit dependence of the costates, $y_3(t)$ on their boundary conditions in each interval, we must make a similar assumption for the form of the cost to go equations so that they will also be independent of the changing boundary conditions. That is, for the interval $t \in [t_j, t_{j+1})$ we define the function

$$V_i(y_1(t),t) \triangleq \frac{1}{2}\{y_1(t)'C_{i1}(t)y_1(t) + y_3(t_{j+1}^-)'C_{i2}(t)y_3(t_{j+1}^-)\} + y_1(t)'C_{i3}(t)y_3(t_{j+1}). \tag{2.44}$$

When evaluated at t_j, with the controls in the interval $[t_j, t_f)$ being the optimal controls defined according to (2.4), this function is then the optimum cost to go, denoted $V_i^*(y_1(t_j), t_j)$. By (2.44) we see that on the interval $[t_j, t_{j+1})$, the cost to go is not only quadratic in y_1, but also has a quadratic term in $y_3(t_{j+1}^-)$ and a cross term in $y_1(t)$ and $y_3(t_{j+1}^-)$.

From the relationship between the costs to go (2.44) and the integrands of the criteria functions (2.43), the differential equations of the coefficient matrices in (2.44) are found to be

$$\dot{C}_{i1} = -\bar{S}_{i1} - C_{i1}\bar{A}_{11} - \bar{A}_{11}'C_{i1} \tag{2.45}$$

$$\dot{C}_{i2} = -\bar{S}_{i2} - 2\bar{A}_{13}'C_{i3} \tag{2.46}$$

$$\dot{C}_{i3} = -\bar{S}_{i3} - C_{i1}\bar{A}_{13} - \bar{A}_{11}'C_{i3} \tag{2.47}$$

where $\bar{A}_{11} = (A_{11} + A_{13}\bar{G}_1)$ and $\bar{A}_{13} = A_{13}\bar{G}_3$.

2.4.3. Boundary Conditions

The boundary conditions for the last interval, that is, at the terminal time, t_f, are

$$\begin{aligned} C_{i1}(t_f) &= K_{if} \\ C_{i2}(t_f) &= 0 \\ C_{i3}(t_f) &= 0. \end{aligned} \tag{2.48}$$

We must also establish appropriate boundary conditions for the remaining intervals. The costs to go must be continuous and therefore

$$V_i(y_1(t_j^-), t_j^-) = V_i(y_1(t_j^+), t_j^+). \tag{2.49}$$

Since the cost to go equations are integrated backwards, we are trying to establish the $C'_{ik}(t_j^-)$ in terms of the $C_{ik}(t_j^+)$ at each j, for each i, and for all k, $k=1,2,3$.

Let us choose

$$C_{i2}(t_j^-)=0 \tag{2.50}$$

$$C_{i3}(t_j^-)=0 \tag{2.51}$$

for all j and for all i. So now we must simply find $C_{i1}(t_j^-)$ in terms of the $C_{ik}(t_j^+)$ for $k=1,2$, and 3.

Due to their interrelatedness, we must simultaneously consider solving for the boundary conditions $y_3(t_j^-)$ from (2.18), (2.20) and (2.44) and solving for the $C_{i1}(t_j^-)$ in terms of the $C_{ik}(t_j^+)$, $k=1,2,3$, from (2.49).

To minimize the required computations, it is advantageous if $y_3(t)$ is broken up

$$y_3 \triangleq \begin{bmatrix} y_3^1 \\ --- \\ y_3^2 \end{bmatrix} \triangleq \begin{bmatrix} \lambda \\ --- \\ p_1 \\ \vdots \\ p_m \end{bmatrix}. \tag{2.52}$$

The derivation of the boundary conditions for the jth interval $[t_j,t_{j+1})$ proceeds as follows. From (2.29), (2.44), (2.50) and (2.51)

$$y_3^2(t_{j+1}^-) = \left.\begin{bmatrix} C_{11} \\ C_{21} \\ \vdots \\ C_{m1} \end{bmatrix}\right|_{t_{j+1}^-} y_1(t_{j+1}) \tag{2.53}$$

and

$$y_3^1(t_{j+1}^-) = [C_{01}y_1 - \overline{C}(F_1y_1 + F_3y_3)]\Big|_{t_{j+1}^-}$$
$$= [C_{01}y_1 - \overline{C}(F_1y_1 + F_3^1y_3^1 + F_3^2y_3^2)]\Big|_{t_{j+1}^-} \tag{2.54}$$

where $\overline{C} = [C'_{11} \vdots C'_{21} \vdots \cdots \vdots C'_{m1}]$ and where F_3 is broken up into $F_3 = [F_3^1 \vdots F_3^2]$ with F_3^1 and F_3^2 having dimensions which correspond to y_3^1 and y_3^2. By substituting (2.53) into (2.54), equation (2.54) becomes

$$y_3^1(t_{j+1}^-) = [C_{01}y_1 - \overline{C}(F_1y_1 + F_3^1y_3^1 + F_3\overline{C}'y_1)]\Big|_{t_{j+1}^-}$$

so

$$y_3^1(t_{j+1}^-) = [(I + \overline{C}F_3^1)^{-1}(C_{01} - \overline{C}(F_1 + F_3^2\overline{C}'))y_1]\Big|_{t_{j+1}^-} \tag{2.55}$$

Combining (2.53) and (2.55) defines D_{j+1}

$$y_3(t_{j+1}^-) = D_{j+1}y_1(t_{j+1}) \tag{2.56}$$

where

$$D_{j+1} = \left.\begin{bmatrix} (I + \overline{C}F_3^1)^{-1}(C_{01} - \overline{C}(F_1 + F_3^2\overline{C}')) \\ \hline \overline{C}' \end{bmatrix}\right|_{t_{j+1}^-} . \tag{2.57}$$

By breaking up y_3 as in (2.52) we need only invert a matrix of dimension n, the system dimension, to obtain D_{j+1}. Otherwise we would have had to invert a matrix of dimension n(m+1).

To find the $C_{i1}(t_{j+1}^-)$ we also need a relationship between $y_3(t_{j+1}^-)$ and $y_1(t_j)$. That is, from (2.40) and (2.56) we can find

$$y_3(t_{j+1}^-) = E_jy_1(t_j) \tag{2.58}$$

where

$$E_j = D_{j+1}(I - H_3(t_{j+1}^-))^{-1}H_1(t_{j+1}^-). \tag{2.59}$$

So, from (2.44), (2.49), (2.50), (2.51) and (2.58)

$$C_{i1}'(t_j^-) = C_{i1}(t_j^+) + E_j'C_{i2}(t_j^+)E_j + 2C_{i3}(t_j^+)E_j. \tag{2.60}$$

We now have all of the required boundary conditions. The cost to go boundary conditions are (2.48), (2.50), (2.51) and (2.60) and the costate boundary conditions are (2.56) or (2.58).

2.4.4. Solution of the Cost to Go Equations

In each interval, we do not need the cost to go for all $t \in [t_j, t_{j+1})$ but rather we only need the value at the initial boundary, i.e., we only need to solve for the $C_{ik}(t_j)$ in terms of the $C_{i1}(t_{j+1}^-)$.

The cost to go equations, (2.45) through (2.47), are the same for each interval and only the boundary conditions change. In order to avoid resolving these equations in each interval, we will assume a functional dependence of the cost to go matrices on their boundary conditions, similar to the technique used on the costates. Since the cost to go equations are linear, we can find such a functional dependence. It will be independent of the changing boundary conditions and can therefore be presolved. The solution of the function will be valid for each interval.

For notational convenience, we will "stack" the columns of the cost to go matrices so that the matrix equations (2.45) through (2.47) can be written as vector equations. Let $\tilde{c}_{ik}$ be the vector corresponding to the matrix C_{ik}. Define $\tilde{c}_i$ as

$$\tilde{c}_i \triangleq \begin{bmatrix} \tilde{c}_{i1} \\ \tilde{c}_{i2} \\ \tilde{c}_{i3} \end{bmatrix}. \tag{2.61}$$

Then (2.45) through (2.47) can be rewritten as

$$\dot{\tilde{c}}_i = \tilde{A}_i\tilde{c}_i + \tilde{b}_i \tag{2.62}$$

where the matrix $\tilde{A}_i$ and the vector $\tilde{b}_i$ are known from the coefficient matrices of (2.45) through (2.47). We can now solve for the functional dependence of the solution of (2.62) in the jth interval on the boundary condition $\tilde{c}_i(t_{j+1}^-)$. Actually, since $\tilde{c}_{i2}(t_{j+1}^-) = 0$ and $\tilde{c}_{i3}(t_{j+1}^-) = 0$, we need only assume dependence of the solution on $\tilde{c}_{i1}(t_{j+1}^-)$, i.e., for $t \in [t_j, t_{j+1})$ assume

$$\tilde{c}_i(t) = M_i(t)\tilde{c}_{i1}(t_{j+1}^-) + d_i(t). \tag{2.63}$$

From (2.62) and (2.63) it follows that

$$\dot{M}_i = \tilde{A}_i M_i, \qquad M_i(t_{j+1}^-) = \begin{bmatrix} I \\ --- \\ 0 \end{bmatrix} \tag{2.64}$$

$$\dot{d}_i = \tilde{A}_i d_i + \tilde{b}_i, \quad d_i(t_{j+1}^-) = 0 \tag{2.65}$$

where the dimension of the identity matrix in $M(t_{j+1}^-)$ is the same as the dimension of $\tilde{c}_{i1}$.

If the system is time invariant and if the sampling rate is constant then (2.64) and (2.65) need be solved only once over one sampling interval. In fact, only the value of $M_i(t_j^+)$ and $d_i(t_j^+)$ need be stored since we only need $\tilde{c}_i(t_j^+)$ in terms of $\tilde{c}_{i1}(t_{j+1}^-)$. That is

$$\tilde{c}_i(t_j^+) = M_i(t_j^+)\tilde{c}_{i1}(t_{j+1}^-) + d_i(t_j^+) \tag{2.66}$$

where $M_i(t_j^+)$ and $d_i(t_j^+)$ are the same for all j.

Due to the relationship (2.63), we will not have to solve the cost to go equations (2.45) through (2.47) repeatedly for each sample interval but need only plug into (2.66).

2.4.5. Summary of Algorithm

We will now summarize the required calculations in the following flow chart. The major steps and reference to the related equations are given in the order in which they must be computed.

All integrations are performed over only one sample interval if the system is time invariant.

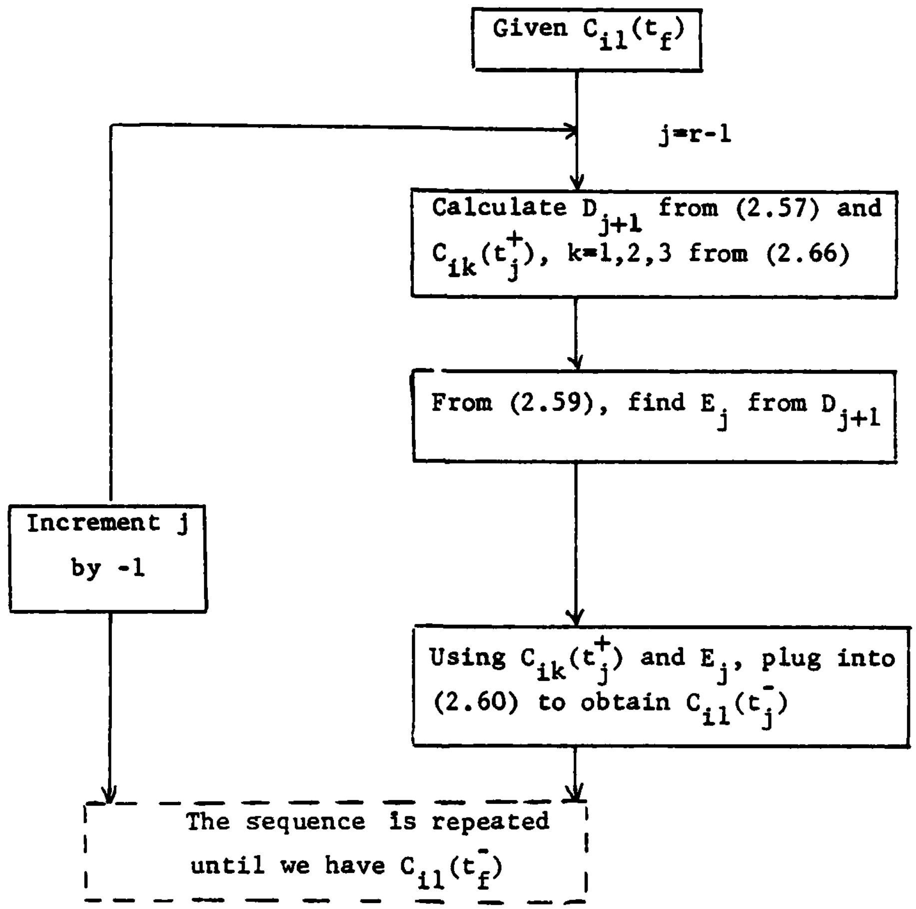
integrate (2.32) through (2.37) to find the G and F matrices

integrate (2.41) and (2.42) to find the H matrices

integrate (2.64) and (2.65) to find the matrices $M_i(t_j^+)$ and the vectors $d_i(t_j^+)$

Recall that $M_i(t_j^+)$ and $d_i(t_j^+)$ are invariant with respect to j for a time invariant system

Going backwards from $j = r-1$ to $j = 1$, beginning with the known $c_{i1}(t_f)$ from (2.48), the following calculations must be done for each j in order to obtain the boundary conditions for each interval.



2.4.6. Implementation

The controls can now be implemented forward in time. They are found by (2.19), (2.22), the definitions of y_3, i.e., $y_3 = (\lambda' \,\vdots\, p_1' \,\vdots\, \cdots \,\vdots\, p_m')'$, and $y_1 = x$, and the evolution of $y_3(t)$ in each interval, $t \in [t_j, t_{j+1})$ given by

$$y_3(t) = P(t)y_1(t_j) \tag{2.67}$$

where

$$P(t) = [\bar{G}_1(t)(H_1(t) + H_3(t)E_j) + \bar{G}_3(t)E_j] \tag{2.68}$$

which is derived from (2.39), (2.40) and (2.58).

If $P(t)$ is broken up as

$$P(t) = \begin{bmatrix} P_o(t) \\ P_1(t) \\ \vdots \\ P_m(t) \end{bmatrix}$$

where each block $P_i(t)$ is n by n, then the ith control during the jth interval is

$$u_i(t) = -R_{ii}^{-1}B_i'P_i(t)x(t_j).$$

As outlined above, there are a number of equations to be integrated, some of which are of large dimension. These integrations, however, are done once only and are performed over a period equal to the length of only one sample interval. Thus, as the number of samples taken increases, the computational burden is reduced. Computationally the only limiting factor which prohibits us from allowing the length of the sample intervals to become arbitrarily small is the corresponding increase in the number of matrix inversions which must be performed at the sampling times in order to generate the required boundary conditions for each interval. That is, as the period of integration becomes smaller, these matrix inversions will tend to become the dominant computational burden. The matrix inversions present another difficulty since, in general, we are unable to guarantee their existence.

2.4.7. Comparison of Techniques

The first technique discussed at the beginning of this section is a method for converting the problem of repeatedly solving m coupled matrix

Riccati equations to that of solving m uncoupled matrix Riccati equations providing significant computational savings. These equations, however, must still be solved repeatedly for each sample interval with only a change in the boundary conditions.

The second approach requires a set of linear and Riccati equations to be solved once only over a horizon which is the length of only one sample interval. The computational advantage of this second technique is due to the fact that the integrations are performed over only one sample interval which is, in general, considerable shorter than the time horizon of the original problem.

The second approach has an advantage over the first approach due to the fact that the equations which are to be solved in the second technique are solved only once for a period equal to one sample interval while the equations to be solved in the first technique must be solved repeatedly during each sample interval. However, for a sufficiently small sample interval it has been observed that the matrix inversions needed to generate the boundary conditions in the second technique can become a dominant factor. Therefore, the advantage shifts to the first technique for the case of decreasing sample interval.

2.5. Conclusions

In this chapter, a sampled data equilibrium Stackelberg strategy has been considered. The advantages of the formulation can be seen by considering certain characteristics of the continuous time Stackelberg problem. The linear quadratic, continuous time, closed loop Stackelberg problem results in a solution, if it exists, in which the controls are non-linear functions of the state. Furthermore, the Stackelberg solution for general dynamic games does not, in general, satisfy the principle of optimality. The principle of optimality can be imposed for discrete time games but the procedure for doing this for general continuous time games has not been established.

The sampled data equilibrium Stackelberg solution results in linear control laws for the linear quadratic case. The advantage of linear control laws is that they are quite simple to implement.

In deriving the sampled data equilibrium Stackelberg solution we have been able to obtain considerable computational savings. That is, rather than performing integrations over the entire time horizon of the original problem, we are able to imbed the subproblems of each sample interval into a more general formulation, the solution of which requires integrations over a period equal to the length of only one sample interval. The computational technique, an application of invariant imbedding developed for the particular case of a Stackelberg strategy and the type of boundary conditions peculiar to it, is quite useful for many problems, in particular for a variety of sampled data formulations.

CHAPTER 3

ON THE EXISTENCE OF STABILIZING SOLUTIONS FOR THE STACKELBERG STRATEGY

3.1. Introduction

In the previous chapter we developed an effective method for the coordination of the decentralized control of a large system by imposing a form of the Stackelberg strategy and exploiting certain characteristics of the strategy. The Stackelberg strategy, as considered in Chapter 2, is one of many forms in which it might arise. Generally it is of interest either as a control strategy to be imposed on a given problem, such as for coordination purposes, or it may arise naturally wherever a precedence relationship exists among the controllers.

While prescriptive approaches to the design of controllers which satisfy the Stackelberg strategy have been developed for many forms of the strategy, little is known about the existence of such control laws or if the system under their control will be stabilized. In this chapter, we will address the problem of the existence of stabilizing solutions for controller, which are obtained according to a Stackelberg strategy.

In order to consider a simple form of the Stackelberg strategy we will examine the problem of a linear system being controlled by two controllers where the control laws are constrained to be in the form of linear, time-invariant state feedback. The cost functions are assumed to be defined over an infinite horizon. It is known [6] that in general there is no optimal <u>linear</u> control law for the leader so we consider the problem in which the leader's cost function is modified in order to average out the dependence of the solution on the initial condition. This formulation is well posed with respect to linear solutions. The necessary conditions for this problem have been derived in [6].

Our interest in the problem is in finding out if there exists a stabilizing solution and if so, presenting conditions under which a stabilizing solution can be guaranteed. This particular form of the Stackelberg strategy, i.e., linear state feedback, is considered because it allows us to use concepts from linear systems theory in approaching the problem.

In Section 3.2 we will introduce the concepts needed to establish the main result which is presented in Section 3.3.

3.2. Background

It will be assumed throughout that we are dealing with a linear time invariant system and linear time invariant control laws.

3.2.1. Controllable subspaces

For the multi-controller case, the individual controllable subspaces are not invariant with respect to feedback. For example, for the two controller case

$$\langle A+B_1F_1 | \mathcal{B}_2 \rangle \neq \langle A | \mathcal{B}_2 \rangle$$

in general, where $\mathcal{B}_2 \triangleq \mathcal{R}(B_2)$ and where

$$\langle A | \mathcal{B} \rangle \triangleq \mathcal{B} + A\mathcal{B} + \cdots + A^{N-1}\mathcal{B}$$

where N is the dimension of the system and $\mathcal{B} = \mathcal{R}(B)$. So the controllable subspace of one controller can be altered by feedback by another controller.

The jointly controllable subspace

$$\langle A | \mathcal{B}_1 \rangle + \langle A | \mathcal{B}_2 \rangle$$

say, is invariant with respect to feedback. This is true for any number of controllers.

For the two DM case we can denote

$$R_i(F_j) \triangleq \langle A + B_j F_j | \mathcal{B}_i \rangle \quad i \neq j, \quad i = 1,2$$

and the following is readily verified.

Lemma 3.1: For $i=1,2$ the subspace R_i depends on F_j, $j \neq i$, and does not depend on F_i.

Thus the notation $R_i(F_j)$, $j \neq i$ is justified.

If we define $\bar{R}_i$ to denote the space perpendicular to R_i then, for given F_1 and F_2;

$R_i(F_j)$ is the smallest $(A+B_jF_j)$-invariant subspace containing $\mathcal{R}(B_i)$, $i \neq j$, $i=1,2$.

$\bar{R}_j(F_i)$ is the largest $(A+B_iF_i)'$-invariant subspace contained in $\mathcal{N}(B_j')$, $i \neq j$, $i=1,2$, where $\mathcal{N}(A) \triangleq$ null space of A.

The following subspace definitions will be useful. The system triple (A,B_1,B_2) uniquely determines the subspaces defined as follows;

R_i^* : largest subspace $\mathscr{S}$ such that $\mathscr{S} \subset R_i(F_j)$ for all F_j, $j \neq i$.

The R_i^* can be thought of as the greatest lower bound (in the sense of subspace inclusion) for the set of subspaces $R_i(F_j)$ over all F_j. The definition of these subspaces is invariant with respect to feedback, i.e., they are the same whether we consider the system (A,B_1,B_2) or $((A+B_1F_1+B_2F_2),B_1,B_2)$ for any F_1,F_2.

3.2.2. Criterion Subspaces

For the quadratic criterion function

$$J_i = \tfrac{1}{2} \int_{t_o}^{\infty} (x'Q_i x + \sum_{j=1}^{m} u_j' R_{ij} u_j)dt$$

and the system

$$\dot{x} = Ax + \sum_{j=1}^{m} B_j u_j \qquad x(t_o) = x_o$$

$$u_i = F_i x.$$

the criterion subspace is defined

$$\mathcal{J}_i(F_j; j=1,\ldots,m) \triangleq \langle \bar{A}' | \mathcal{C}_i' \rangle + \sum_{j=1}^{m} \langle \bar{A}' | \mathcal{F}_{ij}' \rangle$$

where

$$\bar{A} \triangleq (A + \sum_{i=1}^{m} B_i F_i)$$

$$\mathcal{C}_i' \triangleq \mathcal{R}(C_i')$$

$$\mathcal{F}_{ij}' \triangleq \mathcal{R}(F_j' R_{ij}^{1/2})$$

$$C_i' C_i = Q_i \geq 0, \quad R_{ij} \geq 0, \quad R_{ii} > 0 \qquad i,j = 1,\ldots,m.$$

If $x_o \in \mathcal{J}_i$, $x_o \neq 0$, then $J_i > 0$ (possibly infinite). We say that the subspace $\mathcal{J}_i$ is observable through the criterion function J_i.

The criterion subspaces have the following property.

<u>Lemma 3.2</u>:

$$\overset{m}{\underset{i=1}{+}} \mathcal{J}_i^{o\ell} \subseteq \overset{m}{\underset{i=1}{+}} \mathcal{J}_i(F_j;\ j=1,\ldots,m)$$

for all F_j and where $\mathcal{J}_i^{o\ell}$ is the open loop $\mathcal{J}_i$ with $F_j \equiv 0$, $j=1,\ldots,m$.

The proof of this follows in a straightforward way from Theorem 3.6 of [22].

Lemma 3.2 tells us that observability is preserved under feedback.

Another direct consequence of Theorem 3.6 of [22] is the following:

<u>Lemma 3.3</u>: If for some i, the R_{ij}, $j=1,\ldots,m$ are all positive-definite, $R_{ij} > 0$, and the system is detectable through J_i^{ℓ}, i.e., the open loop J_i when all feedback gains equal zero, then the system will remain detectable through $J_i(F_j;\ j=1,\ldots,m)$ for all F_j.

If we define $\bar{\mathcal{J}}_i$ to be the subspace perpendicular to $\mathcal{J}_i$ then, for given F_1 and F_2, in the two controller case, for $R_{ij}=0$, $j\neq i$,

$\mathcal{J}_1(0,F_2)$ is the smallest $(A+B_2F_2)'$-invariant subspace containing $\mathcal{R}(C_1')$

and

$\bar{\mathcal{J}}_2(F_1,0)$ is the largest $(A+B_1F_1)$-invariant subspace contained in $\mathcal{N}(C_2)$.

The subspaces $\mathcal{J}_2(F_1,0)$ and $\bar{\mathcal{J}}_1(0,F_2)$ can be defined similarly.

At this point, we need to be familiar with the following concepts of (A,B)-invariant subspaces [22].

A subspace $\mathcal{V}$ is (A,B)-invariant if and only if there exists an F such that

$$(A+BF)\mathcal{V}\subseteq\mathcal{V}.$$

If S is a set of subspaces then the supremal subspace $\mathcal{V}^*$ of the set S is defined as the subspace $\mathcal{V}$ such that $\mathcal{V}\in S$ and for every $\tilde{\mathcal{V}}\in S$, $\tilde{\mathcal{V}}\subseteq\mathcal{V}$. If the supremal subspace exists, it is unique.

For some subspace Q, the set of all (A,B)-invariant subspaces contained in Q always has a supremal element. With this background we can make the following observation.

<u>Lemma 3.4</u>: There exists unique $\bar{\mathcal{J}}_1^*$ and $\bar{\mathcal{J}}_2^*$ such that

$$\bar{\mathcal{J}}_1(F_1,F_2)\subset\bar{\mathcal{J}}_1^* \text{ for all } F_1,F_2,$$

i.e.,

$$\bar{\mathcal{J}}_2(F_1,F_2)\subset\bar{\mathcal{J}}_2(F_1,0)\subset\bar{\mathcal{J}}_2^*$$

$$\bar{\mathcal{J}}_1(F_1,F_2)\subset\bar{\mathcal{J}}_1(0,F_2)\subset\bar{\mathcal{J}}_1^*$$

where $\bar{\mathcal{J}}_j^*$ is the supremal (A,B_i)-invariant subspace contained in $\mathcal{N}(C_j)$, $j\neq i$, $i=1,2$.

Corresponding to these supremal $\bar{\mathcal{J}}_i^*$ are infimal subspaces $\hat{\mathcal{J}}_i^*$ for which

$$\mathcal{J}_i^* \subset \mathcal{J}_i(F_1,F_2) \quad \text{for all } F_1 \text{ and } F_2.$$

A particular case that will be of interest is as follows:

If, say, $R_{12} > 0$ (positive-definite) then

$$\bar{\mathcal{J}}_1(F_1,F_2) \subset \bar{\mathcal{J}}_1(0,F_2) \subset \bar{\mathcal{J}}_1(0,0) \equiv \bar{\mathcal{J}}_1^*.$$

3.3. Existence of Stabilizing Stackelberg Solution

We consider the following problem. For the system

$$\dot{x} = Ax + B_1u_1 + B_2u_2 \qquad x(t_o) = x_o$$

and linear control laws

$$u_i = F_ix \qquad i = 1,2$$

decision maker i wants to choose F_i to minimize the criterion function

$$J_i = \tfrac{1}{2}\int_{t_o}^{\infty}(x'Q_ix + \sum_{j=1}^{2} u_j'R_{ij}u_j)dt$$

where

$$C_i'C_i = Q_i, \quad R_{ii} > 0 \quad i = 1,2, \quad R_{ij} \geq 0 \quad j \neq i$$

and where they act according to the Stackelberg strategy with DM^1 as the leader [6].

It is known in general there is no optimal linear control law for the leader so we consider the problem in which the leader's cost function is modified in order to average out the dependence of the solution on the initial conditions x_o, i.e., we define

$$\tilde{J}_1 = \underset{x_o}{E}\{J_1\}$$

with

$$E\{x_o\} = 0 \text{ and } E\{x_o x_o'\} = X_o > 0.$$

The necessary conditions for this problem were derived in [6]. The conditions for the infinite horizon problem are as follows

$$F_2 = -R_{22}^{-1}B_2'K_2 \qquad , K_2 = M_2$$

$$\bar{A}'K_2 + K_2\bar{A} + K_2S_{22}K_2 + F_1'R_{21}F_1 + Q_2 = 0$$

$$\bar{A}'M_1 + M_1\bar{A} + K_2S_{12}K_2 + F_1'R_{11}F_1 + Q_1 = 0$$

$$N_2\bar{A}' + \bar{A}N_2 - S_{22}M_1N_1 - N_1M_1S_{22} + S_{12}M_2N_1 + N_1M_1S_{12} = 0$$

$$N_1\bar{A}' + \bar{A}N_1 + X_o = 0$$

$$R_{21}F_1N_2 + R_{11}F_1N_1 - B_1'(M_2N_2 + M_1N_1) = 0$$

where the last equation is solved for F_1.

We would like to answer the question: Under what conditions can we guanratee that there will exist a solution (F_1^*, F_2^*) such that the resultant system

$$\dot{x} = (A + B_1F_1^* + B_2F_2^*)x$$

is asymptotically stable?

For a given F_1, the follower is faced with a conventional optimization for which it is known [23] that if the triple $(C_2, A+B_1F_1, B_2)$ is stabilizable and detectable, then there will exist a unique optimal F_2^* and that the system matrix $(A+B_1F_1+B_2F_2^*)$ will be stable. So, we ask under what conditions does there exist an optimal F_1^* and if it exists, under what conditions will it be chosen such that $(C_2, A+B_1F_1^*, B_2)$ will be stabilizable and detectable?

Theorem 3.1: Existence of stabilizing solution.

We assume that the system is jointly controllable

$$R_1(0) + R_2(0) = R^N$$

and we assume that $R_{12} > 0$ the leader has a positive definite penalty on the follower's control action. If

i) The system is observable through $\mathcal{J}_1^{\mathcal{O}\ell}$

and

ii) $\bar{\mathcal{J}}_2^* \subseteq R_1^*$

then there exists an optimal F_1^* and an optimal F_2^* and the resultant closed loop system

$$(A + B_1^* F_1^* + B_2^* F_2^*)$$

will be asymptotically stable.

Before proving Theorem 3.1, some preliminary results are needed.

After DM^1 applies feedback, the controllable subspaces are $R_1(0)$ and $R_2(F_1)$. We have assumed joint controllability so

$$R_1(0) + R_2(F_1) = R^N.$$

Note that $(A+B_1F_1)R_2(F_1) \subset R_2(F_1)$, i.e., $R_2(F_1)$ is $(A+B_1F_1)$-invariant.

Define the factor space

$$\tilde{X} = R^N/R_2(F_1).$$

This space is isomorphic to $\hat{R}_1(F_1)$ where $\hat{R}_1(F_1)$ is defined as follows: If $R_o(F_1,F_2) = R_1(F_2) \cap R_2(F_1)$ then let $\hat{R}_i(F_i)$ be any subspace such that

$$R_i(F_j) = R_o(F_1,F_2) \oplus \hat{R}_i(F_i) \qquad j \neq i.$$

Since $\bar{X}$ is isomorphic to $R_1(F_1)$, i.e.,

$$R^N/R_2(F_1) \approx \hat{R}_1(F_1)$$

let

$$\bar{\beta}_1 = (\beta_1 + R_2)/R_2, \quad \beta_1 = \mathcal{R}(B_1)$$

and let A_{11} be the map induced by A on $\bar{X}$. Then,

Lemma 3.5:

$$\langle A_{11} | \bar{\beta}_1 \rangle = \bar{X} \approx \hat{R}_1.$$

Proof: Proposition 1.2 of [22].

Now, corresponding to $\hat{R}_1(F_1)$ and $R_2(F_1)$ there is a basis such that the matrix $(A+B_1F_1)$ will be of the form

$$\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}$$

where

$$A_{22} = (A + B_1F_1)|R_2$$

$$A_{11}P = P(A + B_1F_1)$$

where P is the canonical projection,

$$P : R^N \rightarrow \bar{X}.$$

So from Lemma 3.5 we see that the eigenvalues of A_{11} can be placed arbitrarily by DM^1.

In the sequel, when referring to the system, we mean $(C_2, A+B_1F_1, B_2)$ unless noted otherwise.

We will use the notation $\chi^+(A)$ and $\bar{\chi}(A)$ defined as follows:

$\chi^+(A)$ is the space spanned by the eigenvectors corresponding to the unstable eigenvalues.

$\chi^-(A)$ is the space spanned by the eigenvectors corresponding to the stable eigenvectors.

The following lemmas will also be needed.

Consider the set

$$\mathfrak{F}_1 : \{F_1 | \tilde{J}_1(F_1, f_2(F_1)) \leq k\}$$

where $f_2(f_1)$ represents the reaction of decision maker two to the controls of decision maker one. That is, $f_2(F_1)$ is the implicit mapping defined by the optimization performed by decision maker two.

<u>Lemma 3.6</u>: The mapping $F_2 = f_2(F_1)$ is continuous over the set of F_1 for which the triple

$$(C_2, (A+B_1F_1), B_2)$$

is stabilizable and detectable.

<u>Proof</u>: The mapping $F_2 = f_2(F_1)$ is defined implicitly by the solution for K_2 of the Riccati equation

$$0 = (A+B_1F_1)'K_2 + K_2(A+B_1F_1) - K_2B_2'R_{22}^{-1}B_2K_2 + Q_2 + F_1'R_{21}F_1$$

and

$$F_2 = -R_{22}^{-1}B_2'K_2.$$

The partial derivative of the Riccati equation with respect to its solution K_2 is

$$(I \otimes \bar{A}') + (\bar{A}' \otimes I)$$

where $\otimes$ is the Kronecker product (i.e., $A \otimes B = (a_{ij}B)$) and where $\bar{A} = (A+B_1F_1+B_2F_2)$.

If F_1 is such that the system is stabilizable and detectable then $\bar{A}$ will be stable. A characteristic of Kronecker products is that if $\bar{A}$ is stable then so is

$$(I \otimes \bar{A}') + (\bar{A}' \otimes I)$$

and since it has no eigenvalues $\lambda = 0$, it is nonsingular.

By the Implicit Function Theorem, if a function $f(x,y)$ is differentiable with respect to x and y and if

$$0 = f(\bar{x},\bar{y})$$

where $$f : \chi \times \mathcal{Y} \to \chi$$

then there exists a neighborhood N of $\bar{y}$ in $\mathcal{Y}$ over which a continuous, differentiable (implicit) function $g(y)$ is defined such that

$$0 = f(g(y),y), \qquad y \in N$$

if

$$\left.\frac{\partial f}{\partial x}\right|_{\bar{x}} \text{ is nonsingular.}$$

Therefore the function $F_2 = f_2(F_1)$ is continuous (and differentiable) over the set of F_1 for which the system is stabilizable and detectable.

The set $\mathfrak{F}_1$ is a subset of

$$\mathfrak{F}_2 = \{F_1 | \tilde{J}_1(F_1,F_2) \leq k; \quad \text{for any } F_2\}$$

and

Lemma 3.7: The set $\mathfrak{F}_2$ is a subset of $\mathfrak{F}_3$ for a given k, a sufficiently large f and a sufficiently small $\varepsilon > 0$ where

$$\mathfrak{F}_3 : \{F_1 | \|F_1\| \leq f;\ \mathrm{Re}\{\lambda(\bar{A})\} \leq -\varepsilon,\ \varepsilon>0, \text{ for any } F_2\}.$$

Note that $\mathfrak{F}_3$ is closed and bounded.

Proof: For our case where we have $R_{11} > 0$ and $R_{12} > 0$, Lemma 3.7 follows from Lemma 3.8 which we will prove in detail. (Lemma 3.8 is in a more convenient form to work with.)

Lemma 3.8: For

$$\dot{x} = Ax + Bu$$

$$J = \tfrac{1}{2} \underset{x_o}{E} \{ \int_{t_o}^{\infty} (x'Qx + u'Ru)dt \}$$

$$R > 0, \quad E\{x_o\} = 0, \quad E\{x_o x_o'\} = I$$

$$C'C = Q \geq 0$$

$$(C,A); \quad \text{observable}$$

$$(A,B); \quad \text{stabilizable}$$

$$\mathfrak{F}_A = \{F | J(F) \leq k\}$$

$$\mathfrak{F}_B = \{F | \|F\| \leq f;\ \mathcal{R}e(\lambda(A+BF)) \leq -\varepsilon\}.$$

For a given k, there exists a sufficiently large $f < \infty$ and a sufficiently small $\varepsilon > 0$ such that

$$\mathfrak{F}_A \subset \mathfrak{F}_B.$$

A similar result for the case of $Q > 0$ has been obtained in [25] for output feedback.

Proof of Lemma 3.8: For any F for which J(F) is finite the system will be stabilized and the cost will be

$$J(F) = tr(L)$$

where L satisfies

$$L\bar{A} + \bar{A}'L + Q + F'RF = 0 \tag{3.1}$$

where $\bar{A} = (A+BF)$. $\bar{A} = (A+BF)$.

By taking the norm of equation (3.1), we establish that $F \in \mathfrak{F}_A$ implies that

$$\|F\| \leq f$$

where

$$f = \frac{b}{2a} + \sqrt{\frac{b^2}{4a^2} + \frac{c}{a}} \qquad (3.2)$$

where

$$a \triangleq \min_{\substack{F \\ \|F\| = 1}} \{\|F'RF\|\} > 0$$

$$b \triangleq 2f \cdot \|B\|$$

$$c \triangleq 2f \cdot \|A\| + \|Q\| .$$

It remains to show that $F \in \mathfrak{F}_A$ implies the existence of an $\varepsilon > 0$ such that

$$\mathcal{R}e(\lambda(A+BF)) \leqslant -\varepsilon.$$

Define

$$\bar{Q} = Q + F'RF$$

and

$$\tilde{Q}(F) = \sum_{i=0}^{m-1} (A+BF)^{i'} \bar{Q} (A+BF)^{i}.$$

Notice that

$$\mathcal{R}(\bar{A}^{i'} C') = \mathcal{R}(\bar{A}^{i'} C' C \bar{A}^{i})$$

so

$$\mathcal{R}(\langle \bar{A}' | C' \rangle) = \mathcal{R}(\tilde{Q}(F))$$

and

$$\tilde{Q}(F) > 0.$$

If we pre- and post-multiply equation (3.1) by $\bar{A}^{i'}$ and $\bar{A}^{i}$ respectively, and sum these equations over $i=0,1,\ldots,m-1$, we have

$$\tilde{L}\bar{A} + \bar{A}'\tilde{L} + \tilde{Q} = 0$$

where

$$\tilde{L} \triangleq \sum_{i=0}^{m-1} \bar{A}^{i'} L \bar{A}^{i}.$$

Since $\tilde{Q} > 0$, the set $\|F\| \leqslant f$ is closed and bounded and since the eigenvalues of a matrix depend continuously on the elements of the matrix, there exists a q_{min}

$$0 < q_{min} = \min_{\|F\| \leq F} \min_i (\lambda_i(\tilde{Q})).$$

Notice also that

$$\|L\| \leq \sum_{i=1}^{m} \lambda_i(L) = tr(L)$$

for any $L = L' >$

$$L = L' > 0$$

where

$$\|L\| = \sup_{|x|=1} |Lx| = \lambda_{max}(L).$$

Thus

$$\|L\| \leq J(F) = k$$

for

$$F \in \mathcal{F}_A$$

and

$$\|\tilde{L}\| \leq k \cdot \sum_{i=0}^{m-1} (\|A\| + \|B\| \cdot f)^{2i} \triangleq \ell.$$

If

$$\tilde{L} < \frac{1}{2\varepsilon} \tilde{Q}$$

then

$$\mathcal{Re}\{\lambda(\bar{A})\} \leq -\varepsilon.$$

A sufficient condition for $L < \frac{1}{2\varepsilon} Q$ is that

$$\|L\| < \frac{\lambda_{min}(Q)}{2\varepsilon}$$

therefore for $F \in \mathcal{F}_A$ we have

$$\mathcal{Re}\{\lambda(A + BF)\} \leq -\varepsilon$$

for ε such that

$$0 < \varepsilon < \frac{q_{min}}{2\ell} . \tag{3.3}$$

So, for f as in (3.2) and ε from (3.3) we have

$$\mathcal{F}_A \subset \mathcal{F}_B .$$

The proof of Theorem 3.1 will be done in three parts.

First, we show that, under the conditions of the theorem, there exists a leader's control that will make the system stabilizable for the follower and that in order for J_1 to be finite the leader must choose F_1 such that the follower will stabilize the system.

Second, we show that there exists a leader's control such that the system is detectable by the follower and that for such an F_1, the follower will stabilize the system.

The first two parts will establish that there exists an F_1 such that $\tilde{J}_1$ and J_2 are finite and we establish that the control gains considered by DM^1 can be restricted to a set for which the follower will then be faced with a stabilizable and detectable problem. In the third part we show that the leader's optimization can be considered to be over a closed, bounded set on which the follower's control depends continuously on the leader's control and so the leader's cost function will be continuous in F_1 over this set. These conditions are sufficient to establish the desired results.

<u>Part 1</u>

By Lemma 3.5 we know that there exists an F_1 such that the system is stabilizable by the follower. Now we show that the leader must make the system stabilizable to have a finite $\tilde{J}_1$.

We can express $\tilde{J}_1$ as

$$\tilde{J}_1 = \tfrac{1}{2} E\{\lim_{t\to\infty}(x_o' V(t) x_o)\}$$

where

$$V(t) = \int_0^t e^{s\bar{A}'}(C_1'C_1 + F_1'F_1 + F_2'F_2)e^{s\bar{A}}ds$$

with $\bar{A} = (A+B_1F_1+B_2F_2)$ and, without loss of generality, assume $R_{11} = I$ and $R_{22} = I$. If $\bar{A}$ is unstable let μ be an eigenvalue with $\mathrm{Re}(\mu) \geq 0$ and let x be the corresponding eigenvector.

Then

$$x'Vx = \int_0^t e^{2s\mathrm{Re}(\mu)}(|Cx|^2 + |F_1x|^2 + |F_2x|^2)ds.$$

If $\tilde{J}_1$ is to be finite then this integral must be bounded as $t\to\infty$. For the integral to be bounded, we must have

$$Cx = 0, \quad F_1x = 0, \quad \text{and } F_2x = 0$$

so,

$$C\bar{A}^{i-1}x = \mu^{i-1}Cx = 0$$

$$F_j\bar{A}^{i-1}x = \mu^{i-1}F_jx = 0 \qquad i = 1,\ldots,n$$

$$j = 1,2$$

but this implies that

$$x \in \bar{\mathcal{J}}_1(F_1,F_2).$$

This must be true for all unstable eigenvalues of $\bar{A}$, therefore

$$X^+(\bar{A}) \subset \bar{\mathcal{J}}_1(F_1,F_2)$$

but if condition i) holds then by Lemma 3.3

$$\bar{\mathcal{J}}_1(F_1,F_2) = \phi$$

so

$$X^+(\bar{A}) = \phi$$

which for any F_2 is equivalent to

$$X^+(A + B_1F_1) \subset R_2(F_1).$$

Thus $(A+B_1F_1,B_2)$ is stabilizable by DM^2. So we have shown that if $\tilde{J}_1$ is to be finite then DM^1 must choose F_1 such that $(A+B_1F_1,B_2)$ is stabilizable.

Part 2

There exists an $\bar{F}_1$ such that $Re(\lambda(A|R_1(0)) < 0$ and so $R_1(0) \cap X^+(A+B_1\bar{F}_1) = 0$.

But by condition ii)

$$\bar{\mathcal{J}}_2(\bar{F}_1,0) \subset \bar{\mathcal{J}}_2^* \subset R_1^* \subset R_1(0).$$

Therefore

$$X^+(A+B_1\bar{F}_1) \cap \bar{\mathcal{J}}_2(\bar{F}_1,0) = \phi,$$

i.e., the system is detectable for such an $\bar{F}_1$. The system is also stabilizable for such an $\bar{F}_1$, therefore the follower's resultant control $\bar{F}_2$ will cause the system $(A+B_1\bar{F}_1+B_2\bar{F}_2)$ to be asymptotically stable and $\tilde{J}_1 < \infty$ and $J_2 < \infty$.

Part 3

In the previous sections we have established that there exists an F_1 such that the triple

$$(C_2,(A+B_1F_1),B_2)$$

will be stabilizable and detectable and that in order for the leader to have a finite cost, such an F_1 must be chosen. So, the optimal F_1^*, if it exists, must make the system stabilizable and detectable. It remains to establish that a minimizing control exists. If we can establish that the minimizing control, F_1^*, if it exists, will be contained in a closed bounded set $\mathfrak{F}$ and that the leaders cost function is continuous with respect to F_1 over the set $\mathfrak{F}$, they by the Weierstrass theorem the minimum is attained in $\mathfrak{F}$, i.e., F_1^* exists [24].

From this, Lemma 3.7 follows. From Lemma 3.7

$$\mathcal{J}_1 \subset \mathcal{J}_2 \subset \mathcal{J}_3$$

and the optimization can be done over a closed, bounded set over which the cost function is continuous in F_1. By the Weierstrass theorem an optimal stabilizing solution exists.

3.4. Verification of Conditions

Most of the conditions for the existence of a stabilizing solution can be tested by well established techniques. The controllability of (A,B), where $B = B_1 \vdots B_2$ is a concatenation of B_1 and B_2, is readily checked as well as the positive definiteness of the various matrices. Checking the condition

$$\bar{\mathcal{J}}_2^* \subset R_1^*$$

however deserves some further discussion.

The subspace $\bar{\mathcal{J}}_2^*$ is the supremal (A,B_1)-invariant subspace contained in $\mathcal{N}(C_2)$. Algorithms for calculating a set of vectors which span a supremal subspace have been considered by a number of authors, most notably in [26], where attention is paid to the computation reliable components of supremal subspaces by algorithms whose stability and efficiency can be insured.

The subspace R_1^* is defined as the largest space $\mathcal{V}$ such that $\mathcal{V} \subset R_1(F_2)$ over all possible F_2. That is, R_1^* is the greatest lower bound, in the sense of subspace inclusion, for the set of $R_1(F_2)$. The inclusion of $\bar{\mathcal{J}}_2^*$ within R_1^* is to be tested for and so, although it is not clear how to efficiently calculate R_1^* exactly, any computed subspace of R_1^* for which the inclusion holds is sufficient to establish the desired result. Notice that unlike $\bar{\mathcal{J}}_2^*$, there does not in general exist an F_2 such that $R_1(F_2) = R_1^*$. Relationships that do hold and are useful are

$$\bigcup_{F_1} \hat{R}_1(F_1) \subset R_1^* \equiv \bigcap_{F_2} R_1(F_2).$$

We might consider finding the $\hat{R}_1(F_1)$ of maximum dimension. The maximum dimension of $\hat{R}_1(F_1)$ over all F_1 is unique but there is no unique $\hat{R}_1(F_1)$ with maximum dimension. The calculation of a $R_2(F_1)$ of minimum dimension (not unique), and thus the corresponding $\hat{R}_1(F_1)$, can be done by the index and decomposition algorithm of [27]. If $\bar{\mathcal{J}}_2^*$ is contained in one such maximum $\hat{R}_1(F_1)$ then this is sufficient to establish the result. The union of a finite collection of arbitrarily generated subspaces $\hat{R}_1(F_1)$ might also be considered. As a check, for a finite set of arbitrary F_2 and a candidate space $\mathcal{J} \subset R_1^*$, if

$$\mathcal{J} = \bigcap_{\substack{\text{finite} \\ \text{set of} \\ F_2}} R_1(F_2) \tag{3.4}$$

then $\mathcal{J} = R_1^*$ since the right hand side of (3.4) is an upper bound for R_1^*.

3.5. Conclusions

Conditions have been derived which are sufficient to insure the existence of stabilizing feedback gains which satisfy the Stackelberg strategy. These conditions are sufficient and less restrictive conditions may exist. Although it is possible to insure the existence of a minimizing control for the leader, the computation of an optimal F_1 has not yet been dealt with and requires investigation.

CHAPTER 4

AN EXAMPLE OF THE IMPACT OF THE INFORMATION STRUCTURE

4.1. Introduction

In noncooperative decentralized decision making, the information available to the controllers has a far more significant role than in the case of a centralized, or even decentralized, single objective control problem. In Chapters 4 and 5 we will investigate the Nash strategy and the role that the information structure has in the determination of the controls and the resultant cost incurred by each controller. In Chapter 4 an example is presented and discussed in which a decision maker becomes worse off when more information is made available to him. This demonstrates that more information is not necessarily better and, more generally, it demonstrates that the choice of an information structure must be done in a systematic fashion. In Chapter 5 such a systematic approach is developed.

4.2. An Example

A problem in which there are many controllers, each having a different objective, can be formulated as a differential game with the controllers acting according to a particular strategy. In a decentralized problem, where each controller has different, incomplete information, the information structure can have a significant and sometimes surprising impact on the solution.

We examine a fairly realistic problem of a two-area electric power distribution system in which the two area controllers determine constant output feedback gains according to Nash strategy. An example demonstrates a situation in which one controller is worse off when more information is made available to him.

This phenomenon has been noted previously [28], [29], and [30] for static and one-step dynamic systems where the amount of information is characterized in terms of the statistics of noisy measurements. We consider a differential game where the controllers apply output feedback of perfect measurements. The amount of information is characterized in the following sense. Decision maker i (DM_i) measures $y_i = C_i x_i$ where x is the state of the system and C_i is a matrix of appropriate dimension. We say that $y_i^1 = C_{i1}x$ is more informative than $y_i^2 = C_{i2}x$ if $\mathcal{R}(C'_{i1}) \supset \mathcal{R}(C'_{i2})$ where $\mathcal{R}(A)$: range space of the matrix A, i.e., y_i^1 is composed of the measurements y_i^2 (to within an isomorphic transformation) plus additional linearly independent measurement(s).

The effect of the information structure seems counter-intuitive at first, but will be readily understood once the significance of the "availability" of information is explained in terms of the strategy being employed.

4.3. The System

We consider a two area electric power distribution system with a tie-line interconnection. The model is based on [31]. Each area has a steam plant and is modeled by a fifth order system, the states of which are the deviations of the area frequency, the actuator position and the power outputs of a high pressure turbine, an intermediate pressure turbine and a low pressure turbine. The two subsystems along with the tie-line power flow comprise an eleventh order system. The load disturbance of each area is modeled by a first order system so the combined power and disturbance systems comprise a thirteenth order model.

The model is the two interconnected system model for steam powered plants derived in [31]. The model is a linearization of the system about an operating point, describing the system behavior under real power and frequency variations. The state vector is

x_1 - valve displacement - area one

x_2 - power displacement of high pressure turbine - area one

x_3 - power displacement of intermediate pressure turbine - area one

x_4 - power displacement of low pressure turbine - area one

x_5 - frequency deviation in area one

x_6 - tie-line power flow deviation - from area one into area two

x_7 - valve displacement - area two

x_8 - power displacement of high pressure turbine - area two

x_9 - power displacement of intermediate pressure turbine - area two

x_{10} - power displacement of low pressure turbine - area two

x_{11} - frequency deviation in area two

x_{12} - load disturbance in area one

x_{13} - load disturbance in area two.

The controls are

u_1 - set point adjustment in area one

u_2 - set point adjustment in area two.

In case one, DM_1 measures x_5 and in both cases, DM_2 measures x_{11}. The system can be represented as

$$\dot{x} = \left[\begin{array}{c|c|c|c|c} A_s & -a_1 & 0 & a_3 & 0 \\ \hline a_2 & & -a_2 & 0 & 0 \\ \hline 0 & a_1 & A_s & 0 & a_3 \\ \hline 0 & 0 & 0 & \multicolumn{2}{c}{A_d} \end{array}\right] x + \begin{bmatrix} b \\ 0 \\ \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} u_2 + \begin{bmatrix} 0 \\ \\ \tilde{E} \end{bmatrix} v$$

where

$$A_s = \begin{bmatrix} -2. & 0 & 0 & 0 & 0 \\ 4.75 & -5. & 0 & 0 & 0 \\ 0 & .16667 & -.16667 & 0 & 0 \\ 0 & 0 & 2. & -2. & 0 \\ 0 & .025 & .02333 & .035 & -.1125 \end{bmatrix} \qquad a_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ .08333 \end{bmatrix}$$

$$a_3 = \begin{bmatrix} .023685 \\ 0 \\ 0 \\ 0 \\ -.08333 \end{bmatrix} \qquad a_2 = [0 \mid 0 \mid 0 \mid 0 \mid \; 22.21439] \qquad A_d = \begin{bmatrix} -.01 & 0 \\ 0 & -.01 \end{bmatrix}$$

$$b = \begin{bmatrix} 4. \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad E = \begin{bmatrix} 1. & 0 \\ 0 & 1. \end{bmatrix}.$$

The criterion and covariance matrices are

$$R = 1. \qquad V = \begin{bmatrix} 10^{-6} & 0 \\ 0 & 10^{-6} \end{bmatrix}$$

$$\left.\begin{array}{l} Q_{1_{2,2}} = Q_{2_{8,8}} = .3 \\ Q_{1_{3,3}} = Q_{2_{9,9}} = .28 \\ Q_{1_{4,4}} = Q_{2_{10,10}} = .42 \end{array}\right\} \quad \text{Unit penalty on total area power generation.}$$

$$Q_{1_{5,5}} = Q_{2_{11,11}} = 100.$$

The Nash game will determine the gains of each of the area controllers. We assume that in the steady state, each DM will meet the load demand in his own area, i.e., the steady state power generated in area i is equal to the steady state load in area i. For simplicity, the feedforward gain from the area load disturbance is calculated such that if a step increase in load were to occur then the controller for that area alone would, in the steady state, meet the new demand. Thus these feedforward gains are calculated from algebraic steady state conditions and are not considered as control variables in the Nash calculations.

The problem faced by each of the DMers is to minimize his average steady state cost when the system is subject to constantly varying load disturbances.

The overall system is of the form

$$\dot{x} = Ax + B_1u_1 + B_2u_2 + Ev$$

where

$$x = \begin{bmatrix} \tilde{x} \\ z \end{bmatrix}, \quad A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}, \quad B_i = \begin{bmatrix} B_I \\ 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0 \\ \tilde{E} \end{bmatrix}.$$

The dimensions of these submatrices correspond to the dimensions of $\tilde{x}$ and z, where $\tilde{x}$ is the system state, z is the load disturbance, and v is a white noise process with zero mean and covariance V.

For $i = 1,2$ DM_i has measurement $y_i = C_i x$ and will apply linear output feedback $u_i = -F_i y_i$. DM_i's cost function can be expressed [25]

$$J_i = \lim_{t\to\infty} \tfrac{1}{2} \underset{v}{E}\{x(t)'Q_i x(t) + u_i(t)'R_{ii} u_i(t)\}$$

which, with feedback is

$$J_i = \lim_{t\to\infty} \tfrac{1}{2} \underset{v}{E}\{x(t)'(Q_i + C_i'F_i'R_{ii}F_iC_i)x(t)\}.$$

If we define the matrix

$$S \triangleq \lim_{t\to\infty} \underset{v}{E}\{x(t)x(t)'\}$$

then

$$J_i = \tfrac{1}{2} \operatorname{tr}\{S(Q_i + C_i'F_i'R_{ii}F_iC_i)\}.$$

The feedback gains F_1^* and F_2^* are Nash equilibrium values if

$$J_1(F_1^*,F_2^*) \leq J_1(F_1,F_2^*) \quad \text{for all admissible } F_1 \tag{4.1}$$

$$J_2(F_1^*,F_2^*) \leq J_2(F_1^*,F_2) \quad \text{for all admissible } F_2. \tag{4.2}$$

By application of the matrix minimum principle (32), [33], and [25], the following necessary conditions for the Nash equilibrium output feedback gains are obtained, $i = 1,2$.

$$0 = \frac{\partial J_i}{\partial F_i} = R_{ii}F_iC_iSC_i' - B_i'P_iSC_i' \tag{4.3}$$

$$0 = Q_i + C_i'F_i'R_{ii}F_iC_i + \bar{A}'P_i + P_i\bar{A} \tag{4.4}$$

$$0 = EVE' + \bar{A}S + S\bar{A}' \tag{4.5}$$

where

$$\bar{A} = (A - B_1F_1C_1 - B_2F_2C_2).$$

Each controller's primary concern is to minimize the frequency deviations in his own area. The cost functions are symmetric in the sense that each controller penalizes his own area frequency deviations, his area power generation deviations and his own control actions.

In the example, we compare the Nash equilibrium solutions for two information structures. In case one, each DM measures his own area frequency deviations, and in case two, DM_1 has no measurement available for feedback and DM_2 still measures his own area frequency deviation.

In order to compare solutions for different information structures, there must exist a unique solution for each case. We have established that the Nash equilibrium solutions exist and are unique for this problem by direct numerical calculations of the reaction curves of each controller. In order to make such graphical analysis, it is necessary to restrict the number of measurements available to each controller for feedback. Also, some simplistic assumptions in defining each controller's criterion function are necessary to insure uniqueness of the Nash equilibrium solutions, i.e., the explicit appearance of a penalty on the tie-line power flow deviation in the criterion functions would result in multiple equilibrium solutions under these particular information structures. Although it is not penalized, the steady state tie-line power flow deviation would in fact be zero under constant load disturbances as a result of the constraint that each controller alone must, in the steady state, meet a constant load demand in his own area and the fact that the resultant overall system is stable.

These assumptions regarding the criterion function and the number of available measurements are needed to produce a clear, simple example and are not meant to accurately represent a situation that might be encountered in practice, particularly not for such a small scale system. It is in large scale systems in which such restricted information availability can be expected and in which the impact of the information structure becomes most important.

For comparison we have also calculated the solution for the Stackelberg strategy for the information structure of case one with DM_1 as leader. With the Stackelberg strategy, the controllers do not determine their controls simultaneously, as with the Nash strategy, but rather one controller, the leader, will first determine his control and announce his decision to the other controller, the follower, who will then determine his control knowing what the leader's control will be. The leader, in determining his control, takes into account the follower's subsequent optimization. It is assumed that the leader will not deviate from his announced controls and that the leader knows the follower's cost function and is thus able to calculate the follower's reaction to his controls. For a given information structure, the leader will do at least as well as he would playing according to the Nash strategy. Further details and discussion of the Stackelberg strategy can be found in [2], [8], [7], [34], and [6].

The solutions are shown in Figure 4.1. The reaction curves of the two DM's are plotted (where $u_i = -k_i y_i = -k_i f_i$, f_i is the frequency deviation for area i) and the various solutions are indicated. RC_i indicates DM_i's reaction curve, N1 is the Nash equilibrium solution for case one, N2 is the Nash equilibrium solution for case two, and S indicates the Stackelberg solution with DM_1 as leader.

Table 1 summarizes the various solutions and the related costs.

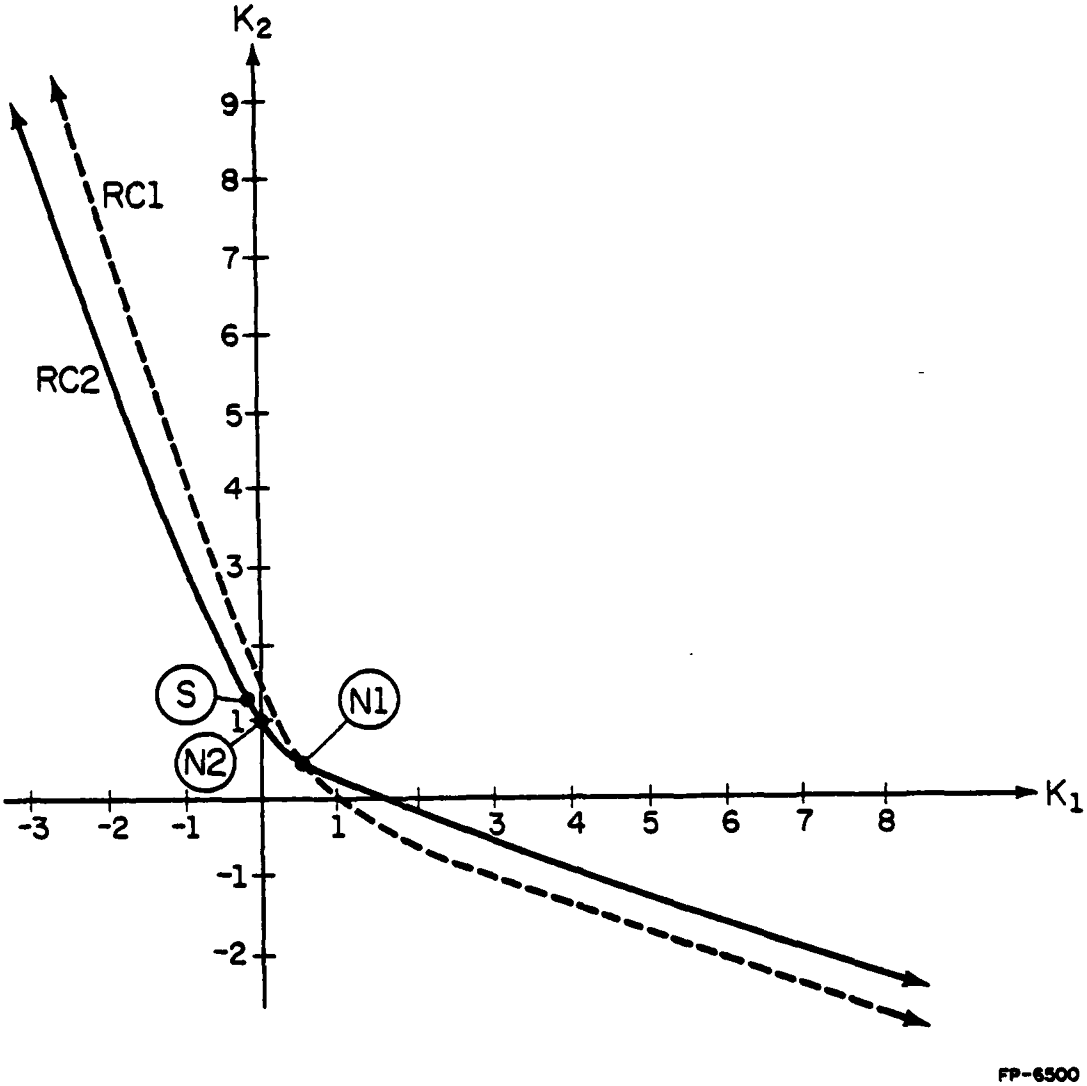


Figure 4.1. Reaction curves and equilibrium points.

Table 1. Resulting Costs for Example

	CONTROL GAINS		COSTS ($\times 10^{-5}$)	
	DM^1	DM^2	DM^1	DM^2
Nash (case 1)	+.40815	+.40815	6.29	6.29
Nash (case 2)	0	+1.02695	3.49	11.4
Stackelberg DM^1: leader	-.199	+1.4171	3.19	14.49

Going down the table we can see that DM_1's cost decreases as we go from case one to case two and from case two to the Stackelberg case. It so happens for this problem that DM_2's cost is increasing as we go down the table. This is not always the case; examples can be constructed in which both DM's are better off when less information is available to one of them and it is also possible that both DMers can have lower cost when using the Stackelberg strategy than when using the Nash strategy [2].

4.4. Discussion

One might reasonably expect that, regardless of the presence of other controllers, if more information is made available to one of the controllers, then that controller would be better off. Why, in case one of the example, could not DM_1 simply ignore the available information, reducing the problem to that of case two? The answer to this, it turns out, is the key to understanding the phenomenon.

The Nash equilibrium conditions, inequalities (4.1) and (4.2) can equivalently be thought of as follows. Each controller is performing an optimization, minimizing his cost function over his entire set of admissible controls, subject to the constraint that the other controller is performing his minimization over his entire set of admissible controls (i.e., over all linear feedback rules for the set of available measurements). In order to have a Nash equilibrium, the optimizations must be consistent; each DM's control must be the optimal over his entire admissible set of controls given that the other DM is applying his Nash control, i.e., inequalities (1) and (2) must hold. Since each controller is assuming that the other controller is optimizing over his entire set of admissible controls, they are each constrained by the consistency requirement to optimize over their own entire set of admissible controls. So, inherent in the Nash inequalities is the constraint that each DM must optimize over all admissible controls; measurements cannot be ignored.

This requirement for consistancy is what constrains the controllers to use the information in a way that could possibly be detrimental to all of the controllers. Is it possible for a controller to avoid this requirement, allowing him to ignore information? Yes, with the Stackelberg strategy, this is accomplished by the leader. By simply allowing for a precedence of decision making, the Stackelberg strategy frees the leader of the requirement that his control must be optimal for the given reaction of the follower. This not only allows the leader to ignore information, if appropriate, but, as demonstrated in the example, he can use the information to his advantage.

In this chapter, we have presented an example which demonstrates that if a dynamic system is to be controlled by more than one controller, and the controllers are acting according to the Nash equilibrium strategy, then a change in the information available to one or more of the controllers can have a surprising effect. In particular, making more information available to one of the controllers does not necessarily bring about improved performance.

In the next chapter we will consider techniques for changing the information structure used in a Nash strategy with the goal of improving some measure of the overall system performance.

CHAPTER 5

INFORMATION DESIGN

5.1. Introduction

In conventional single criterion optimizations, the design of an optimal information structure generally reduces to either the determination of the most informative structure which satisfies certain measurement constraints or to precisely defining the tradeoffs between the cost of acquiring information and the value this information has in terms of its effect on the performance of the control system [35], [36], and [37]. As was demonstrated in the previous chapter, the effect that the information available to one of the controllers has on his performance or on the performance of the other controllers is not quite so self-evident when the decisions are being made according to the Nash strategy. Also, the definition of overall system performance must be made precise if it is to be used in the design of a "better" information structure.

In this chapter we will develop an approach to the design of the information structure that will provide improved performance for the overall system. In order to do this, the design of the information structure for a system in which the controllers are choosing their controls according to a Nash equilibrium strategy must incorporate a precedence relationship, i.e., the Nash equilibrium solution for the controls is done for a given, specified information structure. The information design is not itself a part of the Nash equilibrium conditions but rather is done taking into account the subsequent optimizations being performed according to the Nash strategy. In this sense, the designer of the information structure is behaving as the leader would in a Stackelberg strategy when his control is the information system. The optimization for the information structure must append the subsequent optimizations of the individual DMers.

5.2. The Design Technique

The particular formulation that we will consider is as follows.

A linear system with m controllers acting on it is represented by

$$\dot{x} = Ax + \sum_{i=1}^{m} B_i u_i. \tag{5.1}$$

The ith controller has measurements

$$y_i = C_i x \tag{5.2}$$

and will apply a linear output feedback

$$u_i = -F_i y_i = -F_i C_i x \tag{5.3}$$

in an effort to minimize the cost function

$$J_i = \underset{x_o}{E}\{ \tfrac{1}{2} x_f' K_{if} x_f + \tfrac{1}{2} \int_{t_o}^{t_f} (x' Q_i x + \sum_{j=1}^{m} u_j' R_{ij} u_j) dt\} \tag{5.4}$$

where the expectation over x_o is to remove the dependence of the solution on the initial condition.

The information structure is determined by the output matrices, C_i, in (5.2).

We will develop the information design procedure for the case of linear, static output feedback control (5.3). The extension to the case of the controllers using dynamic compensation of fixed order is straightforward and conceptually equivalent [38].

The Nash equilibrium output feedback gains

$$u_i = -F_i^* y_i$$

are defined as those gains which satisfy the inequalities

$$J_i(F_1^*,\ldots,F_{i-1}^*,F_i^*,F_{i+1}^*,\ldots,F_m^*) \leq J_i(F_1^*,\ldots,F_{i-1}^*,F_i,F_{i+1}^*,\ldots,F_m^*)$$
$$\text{for all } F_i,\ i=1,2,\ldots,m. \tag{5.5}$$

For a given, fixed information structure, (5.2), the necessary conditions for the output feedback gains can be determined by use of the matrix minimum principle [32] and [33]. We will develop these first, in terms of a fixed information structure, and then develop the information design stage.

The cost functions (5.4) can equivalently be written

$$J_i = \tfrac{1}{2}\int_{t_o}^{t_f} \mathrm{tr}\{\bar{Q}_i X\}dt + \mathrm{tr}\{K_{if}X_{t_f}\} \tag{5.6}$$

where

$$\bar{Q}_i = Q_i + \sum_{j=1}^{m} C_j' F_j' R_{ij} F_j C_j$$

and X satisfies

$$\dot{X} = \bar{A}X + X\bar{A}' \tag{5.7}$$

$$X(t_o) = X_o \triangleq E\{x_o x_o'\}$$

$$\bar{A} = (A - \sum_{j=1}^{m} B_j F_j C_j).$$

The Hamiltonian is formed, appending the matrix differential equation (5.7),

$$H_i(X,\Lambda_i,F_j;j=1,\ldots,m) = \tfrac{1}{2}[\mathrm{tr}\{\bar{Q}_i X\} + \mathrm{tr}\{\Lambda_i(\bar{A}X + X\bar{A}')\}], \quad i=1,\ldots,m.$$

From the matrix minimum principle, the minimization of (5.6) with respect to F_i in accordance with the Nash strategy (5.5) yields the following necessary conditions

$$\dot{\Lambda}_i = -\bar{Q}_i - \bar{A}'\Lambda_i - \Lambda_i\bar{A} \tag{5.8i}$$

$$\Lambda_i(t_f) = K_{if}$$

$$\frac{\partial H_i}{\partial F_i} = 0 = R_{ii}F_iC_iXC_i' - B_i'\Lambda_iXC_i' \qquad i = 1,2,\ldots,m. \tag{5.9i}$$

If the feedback gains were constrained to be constant throughout the interval $[t_o,t_f)$ then the condition $\dot{F}_i = 0$ would be appended in the Hamiltonian which would result in equation (5.9i) being replaced by

$$\frac{\partial J_i}{\partial F_i} = \int_{t_o}^{t_f} \frac{\partial H_i}{\partial F_i}\, dt = 0,$$

i.e., for fixed C_i,

$$0 = \frac{\partial J_i}{\partial F_i} = R_{ii}F_iC_iPC_i' - B_i'\Lambda_iPC_i' \tag{5.9i'}$$

where

$$P = \int_{t_o}^{t_f} X(t)dt.$$

For a given information structure (5.2), the control gains must satisfy (5.7), (5.8i), and (5.9i) for i=1,...,m. These equations are a two-point boundary value problem and must be solved iteratively. Since expressions for the gradients are known (5.9i) or (5.9i'), gradient dependent schemes for solving the equations are applicable. Convergence of any approach cannot be assured a priori in that the existence of an equilibrium solution is itself not assured.

We now have a characterization of the behavior of the individual controllers in thier determination of feedback gains for a given information structure. The second phase of the problem, the design of the information structure, can now be developed.

We assume that the information matrices will be chosen to minimize a cost function

$$J_o = \tfrac{1}{2} \underset{x_o}{E}\{ \int_{t_o}^{t_f} (x'Q_o x + \sum_{j=1}^{m} u_j' R_{oj} u_j) dt + x_f' K_{of} x_f \}.$$

It is assumed that this cost function, in some sense, represents an overall system cost which is to be minimized by the choice of the C_i matrices. This might, for example, be a Pareto-optimal cost function agreed on by all of the individual decision makers, i.e., if they agree on the relative importance of their individual costs as expressed by the α_i's then

$$J_o = \sum_{i=1}^{m} \alpha_i J_i, \qquad \alpha_i > 0, \quad \sum_{i=1}^{m} \alpha_i = 1.$$

Alternately, the overall cost function might be a Lyapunov function in which case the information structure is chosen to provide stabilization under the subsequent control actions.

The optimization for the C_i's is done with each C_i held to a fixed allowable maximum rank. Therefore as a special case, one may allow each C_i to attain a maximum rank equal to the dimension of the system thereby admitting full state feedback as an allowable information structure. Note that full state feedback will not necessarily result as being the optimum structure since, as was demonstrated in the previous chapter, more information is not necessarily better. More realistically, there may be only a limited set of measurements available to begin with, e.g., certain states may not be directly measurable at all or only certain states are measurable by certain controllers, allowing for conditions such as geographic separation. These conditions can be treated by assuming the measurements to be in the form

$$y_i = C_i D_i x, \qquad i = 1, \ldots, m$$

where D_i is a fixed matrix and, as before, C_i is the matrix to be determined in the optimization.

Certain special cases are of interest. In particular, if for some i, an initial guess for the information structure is taken to be $C_i = I$, $C_j = 0$, $j \neq i$, then decision maker i is faced with a conventional full state feedback optimization and the remaining controllers are constrained to take no action. This provides a convenient starting point for the iterative calculation of the information structure.

The necessary conditions for the information structure design are presented in the sequel.

For the given J_o, the Hamiltonian is formed to append the necessary conditions which characterize the Nash equilibrium solution for the feedback gains.

$$
\begin{aligned}
H_o &= \tfrac{1}{2}\,\mathrm{tr}\{(Q_o + \sum_{j=1}^{m} C_j'F_j'R_{oj}F_jC_j)X\} \\
&+ \sum_{j=1}^{m} \tfrac{1}{2}\,\mathrm{tr}\{P_{oj}[-Q_j - A'\Lambda_j - \Lambda_jA - \sum_{i=1}^{m}(C_i'F_i'R_{ji}F_iC_i - C_i'F_i'B_i'\Lambda_j - \Lambda_jB_iF_iC_i)]\} \\
&+ \sum_{\ell=1}^{m} \mathrm{tr}\{\beta_{o\ell}'[R_{\ell\ell}F_\ell C_\ell XC_\ell' - B_\ell'\Lambda_\ell XC_\ell']\} \\
&+ \tfrac{1}{2}\,\mathrm{tr}\{\Gamma_o[(A - \sum_{j=1}^{m} B_jF_jC_j)Z + X(A' - \sum_{j=1}^{m} C_j'F_j'B_j')]\}.
\end{aligned}
$$

By the matrix minimum principle the necessary conditions are found to be the following;

$$
\begin{aligned}
\frac{\partial H_o}{\partial F_j} = 0 &= R_{oj}F_jC_jXC_j' \\
&+ \sum_{k=1}^{m}(-R_{kj}F_jC_jP_{oj}C_j' + B_j\Lambda_kP_{ok}C_j') \\
&+ R_{jj}\ {}_{oj}C_jXC_j' - B_j\ {}_{o}XC_j', \qquad j = 1,\ldots,m \qquad (5.10)
\end{aligned}
$$

$$\dot{\Gamma}_o = -\frac{\partial H_o'}{\partial X} = -\{Q_o + \sum_{j=1}^{m} C_j'F_j'R_{oj}F_jC_j$$

$$+ \sum_{\ell=1}^{m} [C_\ell'\beta_{o\ell}'R_{\ell\ell}F_\ell C_\ell + C_\ell'F_\ell'R_{\ell\ell}\beta_{o\ell}C_\ell]$$

$$- \sum_{\ell=1}^{m} [C_\ell'\beta_{o\ell}'B_\ell'\Lambda_\ell - \Lambda_\ell B_\ell\beta_{i\ell}C_\ell] + \bar{A}'\Gamma_i + \Gamma_i\bar{A}\} \tag{5.11}$$

where

$$\bar{A} = (A - \sum_{j=1}^{m} B_jF_jC_j)$$

$$\Gamma_o(t_f) = K_{of}$$

$$\dot{P}_{o\ell} = -\frac{\partial H_o}{\partial \Lambda_\ell} = -\{AP_{o\ell} - P_{o\ell}A' + \sum_{k=1}^{m} [B_kF_kC_kP_{o\ell} + P_{o\ell}C_k'F_k'B_k']$$

$$-B_\ell\beta_{o\ell}C_\ell X - XC_\ell'\beta_{o\ell}'B_\ell'\}, \quad P_{o\ell}(t_o) = 0, \quad \ell = 1,\dots,m \tag{5.12}$$

$$0 = \frac{\partial H_o'}{\partial C_i} = F_i'R_{ii}F_iC_iX - \sum_{j=1}^{m} [F_i'R_{ji}F_iC_iP_{oj} - F_i'B_i'\Lambda_jP_{oj} - P_{oj}\Lambda_jB_iF_i]$$

$$+ \beta_{oi}'R_{ii}F_iC_i + F_i'R_{ii}\beta_{oi}C_iX + \beta_{oi}'B_i'\Lambda_iX - F_i'B_i'\Gamma_oX, \quad i = 1,\dots,m. \tag{5.13}$$

Or, for the optimal time-invariant C_i's, equation (5.13) is replaced by

$$\frac{\partial J_o'}{\partial C_i} = \int_{t_o}^{t_f} \frac{\partial H_o}{\partial C_i}\,dt = 0 \tag{5.13'}$$

which follows from appending the conditions

$$\dot{C}_i = 0.$$

For a given information structure, the feedback gains are found by solving (5.7), (5.8i), and (5.9i) for $i=1,\dots,m$. This provides us with the F_i, X, and the Λ_i, $i=1,\dots,m$. The equations (5.11) and (5.12) are then solved for Γ_o and P_{oi} for $i=1,\dots,m$ where the algebraic equations of (5.10) are solved to eliminate the β_{oi} for $i=1,\dots,m$. The gradient (5.13) or (5.13') can now be evaluated, new C_i's determined, and the process is repeated until convergence is obtained or an adequate improvement in performance is attained.

The amount of computation is quite significant but, fortunately, these computations are done once only and are done by only one decision maker at the time the information structure is being chosen. These computations are transparent to all of the individual controllers since they merely determine their feedback gains for the information structure which, at that time, is determined and fixed. Thus, the precedence relationship of the optimizations isolates the individual controllers from the major computational task of the information structure design. This is a similar effect to the advantage found in the sampled data formulation of Chapter 2. The case in which the available information is restricted, as represented by

$$y_i = C_i \hat{D}_i x$$

is conceptually equivalent and the necessary conditions are developed along identical lines.

For the case in which the problem is defined over an infinite horizon, the necessary conditions for the output feedback gains and for the information structure matrices reduce to a set of algebraic equations.

For this case, the final necessary conditions reduce to the following.

For a given information structure, the output feedback gains must satisfy

$$\bar{A}'\Lambda_i + \Lambda_i\bar{A} + Q_i + \sum_{j=1}^{m} C_j'F_j'R_{ij}F_jC_j = 0 \tag{5.14}$$

$$\bar{A}L + L\bar{A}' + X_o = 0 \tag{5.15}$$

$$0 = \frac{\partial J_i}{\partial F_i} = R_{ii}F_iC_iLC_i' - B_i'\Lambda_iLC_i' \tag{5.16}$$

where

$$\bar{A} = (A - \sum_{i=1}^{m} B_iF_iC_i)$$

for $i = 1,2,\ldots,m$.

The conditions for the optimal information structure are

$$\Gamma\bar{A} + \bar{A}'\Gamma + \sum_{j=1}^{m}\{C_j'\beta_j'(F_{jj}F_jC_j - B_j'\Lambda_j) + (C_j'F_j'R_{jj} - \Lambda_jB_j)\beta_jC_j\}$$
$$+ Q_o + \sum_{j=1}^{m} C_j'F_j'R_{oj}F_jC_j = 0 \tag{5.17}$$

$$0 = R_{oj}F_jC_jLC_j' - B_j'\Gamma LC_j' + R_{jj}\beta_jC_jLC_j' + \sum_{k=1}^{m}\{-B_j'\Lambda_kP_kC_j' + R_{kj}F_jC_jP_kC_j'\} \tag{5.18}$$

which is solved to eliminate the β_j's,

$$0 = AP_\ell - P_\ell A' + \sum_{k=1}^{m}[B_kF_kC_kP_\ell + P_\ell C_k'F_k'B_k'] - B_\ell\beta_\ell C_\ell L - LC_\ell'\beta_\ell'B_\ell' \tag{5.19}$$

$$0 = \frac{\partial J_o'}{\partial C_i} = F_i'R_{ii}F_iC_iL - \sum_{j=1}^{m}[F_i'R_{ji}F_iC_iP_j - F_i'B_i'\Lambda_jP_j - P_j\Lambda_jB_iF_i]$$
$$+ \beta_i'R_{ii}F_iC_i + F_i'R_{ii}\beta_iC_iL + \beta_i'B_i'\Lambda_iL - F_i'B_i'\Gamma L \qquad i = 1,\ldots,m. \tag{5.20}$$

An iterative procedure is possible using the gradient information supplied by (5.20).

Equation (5.18) is used to remove the dependence of (5.17), (5.19), and (5.20) on the β_j's

$$\beta_j = -R_{jj}^{-1}\{R_{oj}F_jC_jLC_j' - B_j\Gamma LC_j' + \sum_{k=1}^{m}[-B_j'\Lambda_kP_kC_j' + R_{kj}F_jC_jP_kC_j']\}(C_jLC_j')^{-1}$$

and if C_j is not full rank, the equation is still solvable since

$$\mathcal{R}(C_jLC_j') = \mathcal{R}(C_j)$$

for $L > 0$, where $\mathcal{R}(\cdot)$ denotes range space.

5.3. An Example

For a simple example to demonstrate the improvement in performance attainable through changes in the information structure, consider the following.

The system

$$\dot{x} = Ax + B_1u_1 + B_2u_2$$

is second order with u_1 and u_2 both scalar

$$A = \begin{bmatrix} -10 & 0 \\ 0 & -10 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} -1 \\ +1 \end{bmatrix} \qquad B_2 = \begin{bmatrix} +1 \\ -1 \end{bmatrix}$$

$$u_i = -f_iy_i = -f_iC_ix.$$

The information structure is given by

$$C_1 = C_2 = \frac{1}{\sqrt{2}} [1 \quad 1].$$

The cost functions

$$J_i = \tfrac{1}{2} E\{\int_0^\infty (x'Q_ix + u_i'R_{ii}u_i)dt\}$$

are specified by

$$Q_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1000 \end{bmatrix}$$

$$Q_2 = \begin{bmatrix} 1000 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$R_{ii} = 1 \qquad i=1,2.$$

For this given nominal information structure, the behavior of the controller is best illustrated by calculating their reaction curves. One to the symmetry of the problem, the reaction curves are symmetric with respect to one another across the 45^o line. Figure 5.1 illustrates the reaction curves where $f_i(f_j)$ denotes controller i's optimal feedback gain as a function of f_j, $j \neq i$, $i = 1,2$. The intersection of the reaction curves is the point which satisfies the Nash inequalities. At the Nash equilibrium point,

$$J_1 = J_2 = 47$$

for which

$$J_o = \tfrac{1}{2} J_1 + \tfrac{1}{2} J_2 = 47.$$

Now let us consider the effect that a change in the information structure can have.

We will consider variations in the information structure parameterized in terms of one parameter as follows:

Let

$$C_1 = [\sin(\theta) \ \cos(\theta)]$$

$$C_2 = [\cos(\theta) \ \sin(\theta)].$$

Notice that by parameterizing the information structure in terms of θ, we are maintaining

$$\| C_i(\theta) \| \equiv 1.$$

Variation of θ is a rotation of the measurement vectors in state space.

The reaction curves and equilibrium points for a few values of θ are shown in Figure 5.2.

In particular, the optimal value of θ is found to be $\theta = -45^o$ for which $J_o = 16.3$. The reaction curves for this case are shown in Figure 5.2c).

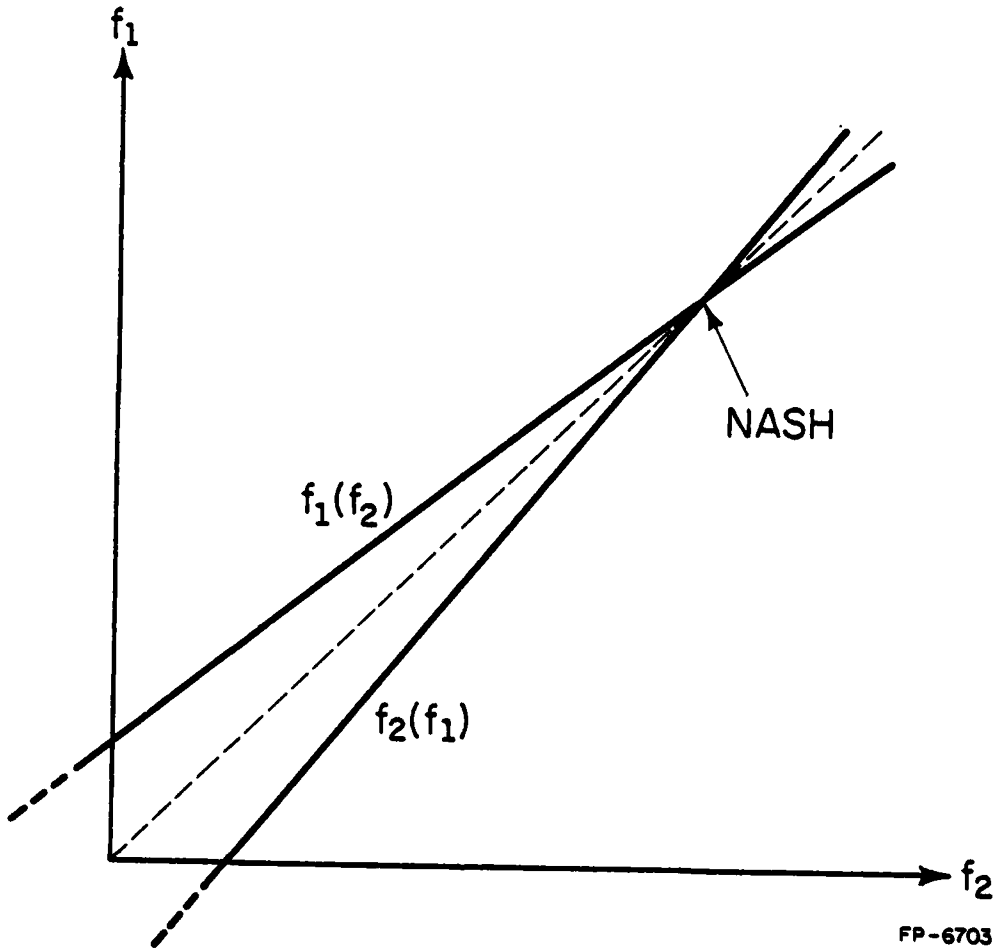


Figure 5.1. Reaction curves for nominal information structure.

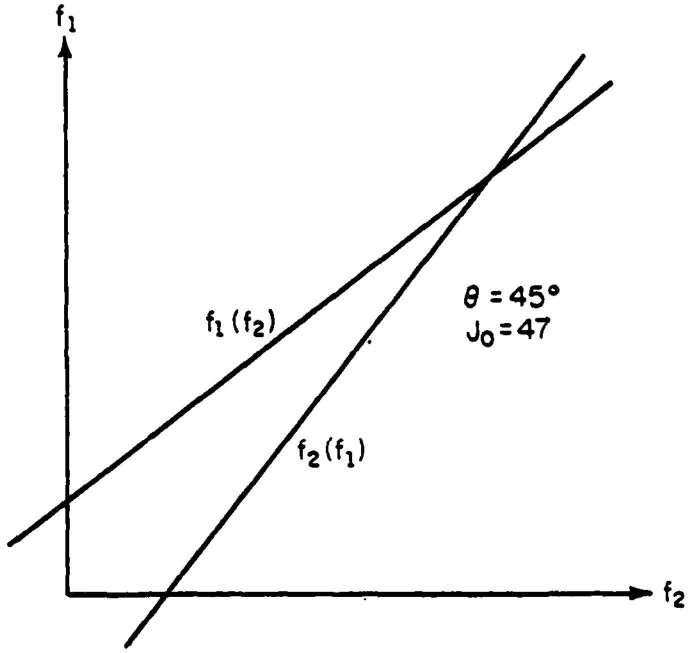


Figure 5.2a. Reaction curves for θ = 45.

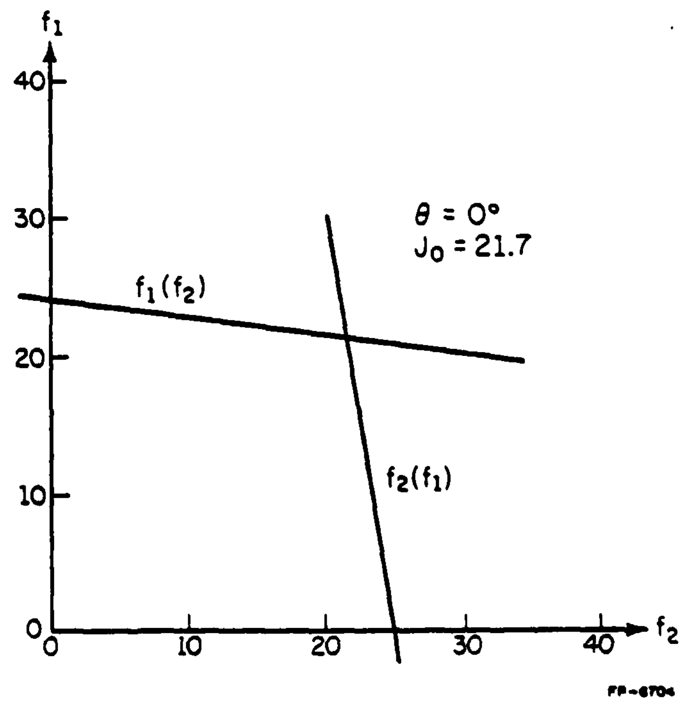


Figure 5.2b. Reaction curves for θ = 0.

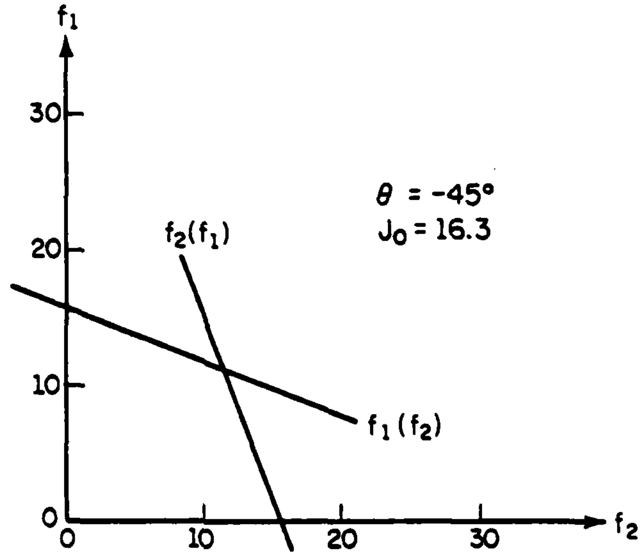


Figure 5.2c. Reaction curves for θ = -45.

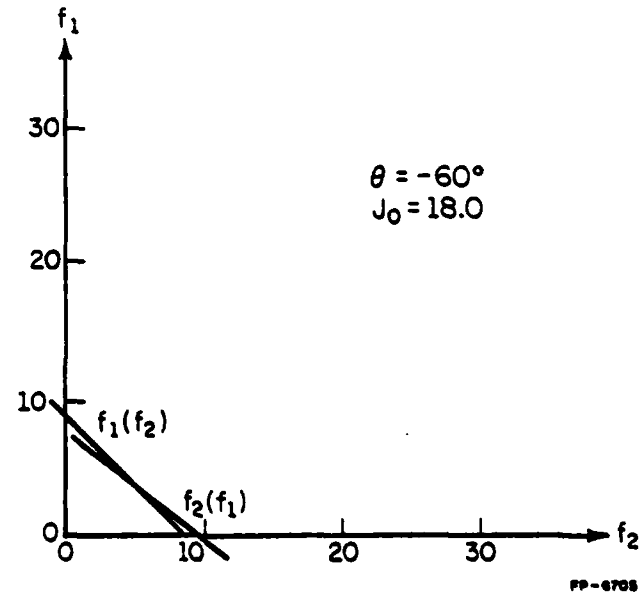


Figure 5.2d. Reaction curves for θ = -60.

For this information structure the costs incurred by each controller and the overall cost are all reduced to approximately one-third of their values for the original nominal information structure which corresponds to $\theta = +45^{\circ}$.

5.4. Conclusions

The optimization for the information structure must append the necessary conditions which characterize the subsequent calculations for the Nash equilibrium solution. The continuity of the Nash equilibrium conditions with respect to the parameters of the information structure is not insured and requires further investigation if any assurances are sought for a well behaved, convergent algorithm. In fact, an algorithm for the calculation of Nash equilibrium output feedback gains for a given information structure with conditions which guarantee convergence would be significant in its own right.

CHAPTER 6

CONCLUSIONS

The decentralized control problem where the individual controllers have different goals has been considered. We have focused on the role of the Stackelberg strategy, particularly for its application for the coordination of many controllers.

Several issues related to the applicability of the strategy have been dealt with, resulting in tractable, efficient algorithms. The structure of the solution to a sampled data formulation has been exploited to obtain particularly efficient solution techniques.

The existence of solutions satisfying the Stackelberg strategy cannot in general be assured a priori. Conditions sufficient for guaranteeing the existence of a solution satisfying a Stackelberg strategy have been developed. These are merely sufficient conditions and there is need for further development.

The impact that the information structure can have on a solution satisfying the Nash strategy has been illustrated by means of an example. The example serves as motivation for the next section in which an approach to the design of the information structure has been developed which exploits the precedence nature of the Stackelberg strategy. The information structure alone is manipulated in an effort to coordinate the subsequent actions of the controllers. As with most coordination schemes using the Stackelberg strategy, the activities of the leader or coordinator are transparent to the individual controllers.

REFERENCES

1. C. I. Chen and J. B. Cruz, Jr., "Stackelberg Solution for Two Person Games with Biased Information Patterns," IEEE Trans. on Automatic Control, Vol. AC-17, No. 6, 1972, pp. 791-798.

2. M. Simaan and J. B. Cruz, Jr., "On the Stackelberg Strategy in Nonzero-Sum Games," J. Opt. Theory Appl., Vol. 11, No. 5, 1973, pp. 533-555.

3. D. Castanon and M. Athans, "On Stochastic Dynamic Stackelberg Strategies," Automatica, Vol. 12, 1976, pp. 177-183.

4. M. Simaan, "Stackelberg Optimization of Two-Level Systems," IEEE Trans. on Systems, Man, and Cybernetics, Vol. SMC-7, July 1977, pp. 554-557.

5. J. B. Cruz, Jr., "Stackelberg Strategies for Multilevel Systems," in Directions in Large Scale Systems, Y. C. Ho and S. K. Mitter, editros, Plenum Publishing Corp., New York, 1976, pp. 139-147.

6. J. Medanic, "Closed-Loop Stackelberg Strategies in Linear-Quadratic Problems," IEEE Trans. on Automatic Control, Vol. AC-23, No. 4, August 1978, pp. 652-657.

7. J. B. Cruz, Jr., "Survey of Nash and Stackelberg Equilibrium Strategies in Dynamic Games," Annals of Economic and Social Measurement, Vol. 4, No. 2, 1975, pp. 339-344.

8. M. Simaan and J. B. Cruz, Jr., "Additional Aspects of the Stackelberg Strategy in Nonzero-Sum Games," J. Opt. Theory Appl., Vol. 11, No. 6, June 1973, pp. 613-626.

9. M. Simaan and J. B. Cruz, Jr., "Sampled Data Nash Controls in Nonzero-Sum Games," Int. J. of Control, Vol. 17, 1973, pp. 1201-1209.

10. M. Simaan and J. B. Cruz, Jr., "On the Solution of the Open Loop Nash Riccati Equations in Linear Quadratic Differential Games," Int. J. of Control, Vol. 18, 1973, pp. 57-63.

11. M. Simaan and J. B. Cruz, Jr., "A Stackelberg Strategy for Games with Many Players," IEEE Trans. on Automatic Control, Vol. AC-18, No. 3, 1973, pp. 322-324.

12. C. Y. Chong and M. Athans, "On the Periodic Coordination of Linear Stochastic Systems," Automatica, Vol. 12, 1976, pp. 321-335.

13. R. Bellman and G. M. Wing, An Introduction to Invariant Imbedding, Wiley-Interscience, New York, 1975.

14. E. D. Denman, Coupled Modes in Plasmas, Elastic Media and Parametric Amplifiers, American Elsevier, New York, 1971.

15. J. E. Prussing, "A Simplified Method for Solving the Matrix Riccati Equation," Int. J. Control, Vol. 15, No. 5, 1972, pp. 995-1000.

16. J. P. Corfmat and A. B. Morese, "Decentralized Control of Linear Multivariable Systems," Automatica, Vol. 12, 1976, pp. 479-495.

17. E. F. Mageirou and Y. C. Ho, "Decentralized Stabilization via Game Theoretic Methods," Automatica, Vol. 13, 1977, pp. 393-399.

18. T. D. Linton, D. R. Fisher, E. C. Tacker, and C. W. Sanders, Jr., "Decentralized Control of an Interconnected Electric Energy System Subject to Information Flow Constraints," Proc. 1973 IEEE Conf. on Decision and Control, pp. 728-730.

19. D. D. Siljak and M. K. Sundareshan, "A Multilevel Optimization of Large-Scale Dynamic Systems," IEEE Trans. on Automatic Control, Vol. AC-21, No. 1, February 1976, pp. 79-84.

20. M. Aoki and M. T. Li, "Controllability and Stabilizability of Decentralized Dynamic Systems," Proc. 1973 JACC, pp. 278-286.

21. M. Aoki, "On the Feedback Stabilizability of Decentralized Dynamic Systems," Automatica, Vol. 8, 1972, pp. 163-173.

22. W. M. Wonham,"Linear Multivariable Control: A Geometric Approach," Lecture Notes in Economics and Mathematical Systems, Springer-Verlag, 1974.

23. V. Kucera, "A Contribution to Matrix Quadratic Equation," IEEE Trans. on Automatic Control, Vol. AC-17, No. 3, 1972, pp. 344-347.

24. D. G. Luenberger, Optimization by Vector Space Methods, John Wiley and Sons, New York, 1969.

25. B.-Z. Kurtaran and M. Sidar, "Optimal Instantaneous Output-Feedback Controllers for Linear Stochastic Systems," Int. J. Control, Vol. 19, No. 4, 1974, pp. 797-816.

26. B. C. Moore and A. J. Laub, "Computation of Supremal (A,B)-Invariant and Controllability Subspaces," IEEE Trans. on Automatic Control, Vol. AC-23, No. 5, October 1978, pp. 783-792.

27. W. M. Wonham and A. S. Morse, "Feedback Invariants of Linear Multivariable Systems," Automatica, Vol. 8, 1972, pp. 93-100.

28. T. Basar and Y. C. Ho, "Informational Properties of the Nash Solutions of Two Stochastic Nonzero-Sum Games," J. Economic Theory, Vol. 11, 1974, pp. 370-387.

29. Y. C. Ho, I. Blau, and T. Basar, "A Tale of Four Information Structures," Lecture Notes in Economics and Mathematical Systems, Vol. 107, Springer-Verlag, 1975.

30. Y. C. Ho and I. Blau, "A Simple Example on Information and Performance," J. of Opt. Theory and Appl., Vol. 11, No. 4, 1973.

31. M. Calovic, "Dynamic State Space Models of Electric Power Systems," Departments of Electrical Engineering and Mechanical Engineering, University of Illinois, 1971.

32. M. Athans, "The Matrix Minimum Principle," Inform. Contr., Vol. 11, Nov.-Dec. 1967, pp. 592-606.

33. P. M. Walsh, "On Symmetric Matrices and the Matrix Minimum Principle," IEEE Trans. on Automatic Control, Vol. AC-22, No. 6, December 1977, pp. 995-996.

34. J. B. Cruz, Jr., "Leader-Follower Strategies for Multilevel Systems," IEEE Trans. on Automatic Control, Vol. AC-23, No. 2, April 1978, pp. 244-255.

35. B. Z. Kurtaran and M. Sidar, "Optimization of the State-Output Relationships for Linear Stochastic Systems," IEEE Trans. on Automatic Control, Vol. AC-19, No. 5, October 1974, pp. 590-592.

36. K. C. Chu, "Designing Information Structures for Quadratic Decision Problems," J. of Opt. Theory and Appl., Vol. 25, No. 1, May 1978, pp. 139-160.

37. L. Meier, J. Peschon, and R. M. Dressler, "Optimal Control of Measurement Subsystems," IEEE Trans. on Automatic Control, Vol. AC-12, No. 5, October 1967, pp. 528-536.

38. W. S. Levine, T. L. Johnson, and M. Athans, "Optimal Limited State Variable Feedback Controllers for Linear Systems," IEEE Trans. on Automatic Control, Vol. AC-16, No. 6, December 1971, pp. 785-792.

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